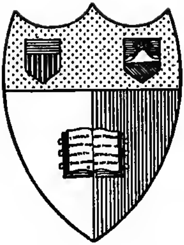
COMMERCIAL ALGEBRA
BOOK II



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WENTWORTH-SMITH MATHEMATICAL SERIES

COMMERCIAL ALGEBRA

BOOK II

BY

GEORGE WENTWORTH
DAVID EUGENE SMITH
AND
WILLIAM S. SCHLAUCH

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PREFACE

This work has been prepared to meet the needs of advanced classes in algebra in commercial high schools. It presupposes one year's work in algebra, and while this may preferably be pursued in the study of Book I, which emphasizes the commercial aspect of the subject as far as seems reasonable, any good course in the first year of algebra will furnish sufficient preparation for the work now in hand.

The most important commercial features of the second year in this kind of algebra are compound interest and its extensive applications in modern business, equation of payments, annuities, amortization, depreciation, bond valuation, and life insurance. In order to be adequately prepared to undertake the study of these features, however, it is necessary to have a thorough working knowledge of logarithms and series. For a good understanding of logarithms it is necessary to introduce a chapter on powers and roots, and this chapter becomes, therefore, the logical point of departure for the year's work. Logarithms naturally lead to a study of the slide rule, an instrument that has recently come into very extensive use in the industrial work of this country, and which is certain to be of large service in checking computations in the problems of commerce.

Looked at from the standpoint of commercial needs, abstract topics like powers, roots, and series assume an importance not found in the ordinary courses in algebra in our high schools. Even a subject like compound interest has little real application as conventionally treated. When related to commercial life, however, compound interest will be seen to touch nearly every kind of investment, and the subject therefore acquires a new value in the eyes of the student. It is the same with subjects like annuities and equation of payments, both of which were discussed in a perfunctory manner in the arithmetics of a generation ago, but with not enough knowledge of mathematics on the part of the students to allow for that thoroughness of treatment which is necessary if one is to make any real use of them. Their value in modern commercial life is becoming more apparent every year, and the student who expects to enter this field of activity needs an equipment which only algebra can give.

To a greater extent than was possible in Book I, the problems are derived from cases actually arising in business. It is desirable, however, to give many applications outside the field of commerce simply as interesting exercises in computation.

Modern business has been placed in the hands of corporations to an extent unthought of a few years ago, and whether it so remains or becomes a function of the state, the capitalization of business and the borrowing of necessary funds is to no small degree a matter of commercial algebra. Amortization, depreciation, and bond valuations are of vital significance in the business life of to-day.

The authors acknowledge with pleasure the courtesy of Professor E. B. Skinner in permitting them to make use of certain mathematical tables prepared originally for his Mathematical Theory of Investment (Ginn and Company, Boston). Students will find it convenient to purchase these tables separately for use in their commercial work.

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COMMERCIAL ALGEBRA

BOOK II

CHAPTER I

POWERS AND ROOTS

1. Purpose of the Chapter. Commercial algebra is concerned with many subjects like compound interest, equation of payments, annuities, amortization, bond valuations, and insurance, which require more elaborate computations than those found in elementary arithmetic. It not only develops shorter methods for the direct cases of these subjects, but it enables the student to obtain solutions for certain converse cases when this is practically impossible by arithmetic. Modern business has no time for the older methods of computation, particularly in its large transactions, and for this reason it adopts many labor-saving methods that are not found in the arithmetic of the schools. Among these methods may be mentioned those of logarithms, the slide rule, quarter squares, the various forms of calculating machines, and elaborate tables. Even to understand and use these methods, without attempting to improve upon them, some further knowledge of algebra is required, and the student has now reached a point where a knowledge of powers and roots is necessary to his further advance.

Some such knowledge has already been obtained in Book I or in some other preliminary work, and it is now proposed to review this knowledge and then to extend it.

2. Product of Powers. Since a^2 means $a \cdot a$, and a^3 means $a \cdot a \cdot a$, we see that

$$a^2 \cdot a^3 = a \cdot a \times a \cdot a \cdot a = a \cdot a \cdot a \cdot a \cdot a = a^5 = a^{2+3}.$$

Furthermore, $a^m = a$ taken m times as a factor,

and $a^n = a$ taken n times as a factor.

Therefore $a^m \cdot a^n = a$ taken $m + n$ times as a factor.

That is, $$\mathbf{a^m \cdot a^n = a^{m+n}}.$$

In multiplying monomials the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.

3. Power of a Power. Since $(a^2)^3$ means $a^2a^2a^2$, or a^6, and $(a^m)^n$ means $a^ma^ma^m \cdots$ to n of the factors a^m, we have

$$\mathbf{(a^m)^n = a^{mn}}.$$

For example, $(a^3)^5 = a^{15}$, $(x^7)^3 = x^{21}$, and so on.

4. Law of Signs in Multiplication. The law of signs in multiplication is already familiar to the student:

Two like signs produce plus; two unlike signs produce minus.

The product of any number of positive factors is positive.

The product of an even number of negative factors is positive; the product of an odd number of negative factors is negative.

Therefore $$-3\,x \cdot (-2\,x^2y) \cdot 5\,x^3y^2 = 30\,x^6y^3,$$
$$2\,x \cdot (-4) \cdot 2\,xy \cdot (-5\,x^2y^2) = 80\,x^4y^3,$$
$$(-7\,x^2)^2 = 49\,x^4,$$
and $$(-4\,a^2)^3 = -64\,a^6.$$

5. Degree of a Term. The number of literal factors in a term is called the *degree of the term.*

Thus a^2 is of the second degree, a^m is of the mth degree, and a^{m+3} is of the $(m+3)$d degree. Similarly, x^2y^3 is of the second degree in x, the third degree in y, and the fifth degree in x and y.

Exercise 1. Products and Powers

Examples 1 to 33, oral

Simplify the following:

1. a^3a^4.

2. b^4b^9.

3. c^6c^7.

4. d^8d^7.

5. e^9e^9.

6. p^5p^{12}.

7. a^4a^5.

8. a^ma.

9. a^ma^p.

10. 0.02^2.

11. 0.02^3.

12. $a^{2x}a^x$.

13. $a^{4x}a^{3x}$.

14. $ab \cdot ab$.

15. $a^3b \cdot a^3b$.

16. $a^3b^3 \cdot a^3b^3$.

17. $p^4q^7 \cdot p^8q^9$.

18. $8\,a^6 \cdot 6\,a^7$.

19. $8\,x^m \cdot 6\,x$.

20. $9\,x^m \cdot 5\,x^n$.

21. $-a^3 \cdot 5\,a^7$.

22. $-2\,a^4 \cdot 6\,a^6$.

23. $4\,a \cdot (-5\,a)$.

24. $4\,a \cdot (-5\,a^2)$.

25. $6\,a^m \cdot (-8\,a^n)$.

26. $2\,ab \cdot (-ab)$.

27. $-ab \cdot 2\,ab$.

28. $x^3y^3 \cdot (-2\,xy)$.

29. $-3\,xy^3 \cdot (-7\,xy)$.

30. $-8\,x^2y^5 \cdot 10\,x^3y^3$.

31. $a^3b^3c^3 \cdot abc$.

32. $(1+r)^2(1+r)^3$.

33. $(1+r)^4(1+r)^7$.

34. If the circumference of a circle is πd, what is the sum of the circumferences of seven circles of diameter d?

35. Find the values of 0.1^2, 0.01^3, 0.001^2, and 0.0001^3, and write the law for the number of decimal places in the power of a decimal.

36. What is the volume of a cube if the edge is $4\,x$?

Simplify the following, merely indicating the powers:

37. $0.02^2 \cdot 0.02^3$.

38. $0.002^2 \cdot 0.002^3$.

39. $-37\,p^x \cdot (-31\,p^y)$.

40. $-90\,p^x \cdot (-21\,p^{2x})$.

41. $(15\,x^2)^3 \cdot (94\,x^3)^3$.

42. $1.02^2 \cdot 1.02^3$.

43. $1.02^3 \cdot 1.02^3$.

44. $1.03^2 \cdot 1.03^3$.

45. $P(1+i)^5(1+i)^2$.

46. $P(1+i)^n(1+i)^2$.

6. Quotient of Powers. Division being the inverse of multiplication, we have the following:

From $a^2 \cdot a^3 = a^5$, we have $a^5 \div a^3 = a^2$;

and from $a^x \cdot a^y = a^{x+y}$, we have $a^{x+y} \div a^y = a^x$,

or, using m instead of $x + y$ and n instead of y,

$$\boldsymbol{a^m \div a^n = a^{m-n}}.$$

The exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.

That is, $-21\,x^9y^4 \div 7\,x^5y = -3\,x^4y^3$.

In algebra, division is usually expressed in the fractional form; thus,

$$\frac{48\,a^5b^4c^3}{-4\,abc} = -12\,a^4b^3c^2.$$

7. Law of Signs in Division. The Law of Signs in division is already familiar to the student:

In division two like signs produce plus; two unlike signs produce minus.

Exercise 2. Quotients of Powers

Examples 1 to 4, oral

Simplify the following:

1. $\dfrac{a^3b^2}{ab^2}$.
2. $\dfrac{m^4n^4}{m^2n^2}$.
3. $\dfrac{x^6y^8}{xy}$.
4. $\dfrac{x^7y^9}{x^2y^3}$.
5. $\dfrac{11.4\,x^7}{3\,x^2}$.
6. $\dfrac{-9.2\,m^4n}{-0.4\,mn}$.
7. $\dfrac{425\,a^4b^3c}{5\,abc}$.
8. $\dfrac{60\,x^4y^4z^4}{-6\,xyz}$.
9. $\dfrac{156\,m^4n^6}{-4\,m^4n}$.
10. $\dfrac{-126\,m^7n^5}{7\,m^5n^5}$.
11. $\dfrac{-135\,x^2y^4z^6}{-5\,x^2yz^5}$.
12. $\dfrac{-325\,m^2n^2p^4}{-25\,mn^2p}$.
13. $\dfrac{1.05^6}{1.05^4}$.
14. $\dfrac{(1+i)^7}{(1+i)^5}$.
15. $\dfrac{(1+i)^n}{(1+i)^2}$.
16. $\dfrac{(1+i)^n}{(1+i)^p}$.

8. Root. Any one of the equal factors whose product is a given expression is called a *root* of the expression.

The root is called a square root, cube root, fourth root, or rth root, according as it is one of the two, three, four, or r equal factors.

The word *root* is used in two different senses in algebra. The other use is to represent the value of the unknown quantity in an equation. Thus the square root of 25 is 5, and the root of the equation $x - 7 = 23$ is 30.

The common symbol for root ($\sqrt{\ }$) is called the *radical sign.*

Thus $\sqrt[4]{625} = 5$, because $625 = 5 \cdot 5 \cdot 5 \cdot 5$. In this case 4 is called the *index* of the root. In $\sqrt[n]{a-b}$, n is the index of the root. In the case of a square root, such as $\sqrt{4}$, the index 2 is not written.

9. Imaginary Number. Since any even power of either a positive or a negative number is positive, we cannot have, in the ordinary sense, an even root of a negative number.

We can merely indicate such roots thus: $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt[4]{-2}$, $\sqrt[6]{-5}$. Such numbers are not needed in commercial algebra, but they have various important applications in the study of physics. Roots of this kind are mentioned at this time simply because the student may meet them in some of his work in equations.

An indicated square root of a negative number is called an *imaginary number.*

For example, $\sqrt{-1}$ and $\sqrt{-4}$ are imaginary numbers. A number of the form $2 + \sqrt{-3}$ or $3 - \sqrt{-2}$ is called a *complex number.*

10. Real Number. A number that does not contain an imaginary number is called a *real number.*

11. Rational Number. An integer or the quotient of two integers is called a *rational number.*

For example, 3, -7, $\frac{2}{3}$, $\frac{5}{2}$, 1.25, and -0.75 are rational numbers.

12. Irrational Number. A real number that is not rational is called an *irrational number.*

For example, $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$ are both irrational numbers. Similarly, such numbers as $-2\sqrt{3}$ and $\frac{2}{3}\sqrt[3]{\frac{1}{2}}$ are irrational.

13. Signs of Roots. Since we know that $(+2)(+2)=4$, and $(-2)(-2)=4$, we see that 4 has two square roots, $+2$ and -2. For the same reason it is evident that

Every positive number has two square roots which have the same absolute values but have opposite signs.

For example, 9 has two square roots, $+3$ and -3. A negative number also has two square roots, but (§ 9) they are imaginary.

In general, every positive number has two real even roots; thus, 16 has two fourth roots, $+2$ and -2, since $(+2)^4=16$, and $(-2)^4=16$. For the sake of completeness it should be stated that 16 has two other fourth roots, but they are imaginaries, $2\sqrt{-1}$ and $-2\sqrt{-1}$.

Furthermore, since $(+2)^3=+8$, and $(-2)^3=-8$, we see that $\sqrt[3]{+8}=+2$, and $\sqrt[3]{-8}=-2$. For the same reason it is evident that

An odd root of a number has the same sign as the number itself.

For example, because $(-2)^5=-32$, we see that $\sqrt[5]{-32}=-2$, and, because $(+2)^5=+32$, we see that $\sqrt[5]{+32}=+2$.

14. Principal Root. The one real root of a number, if it has but one real root, or its real positive root, if it has two real roots, is called the *principal root* of the number.

Thus the principal square root of 4 is $+2$, although there is another square root, -2; and the principal cube root of -27 is -3, although -27 has two other cube roots, both of them being imaginary numbers. In commercial algebra we are concerned only with the principal roots of numbers.

15. Radical Sign and Principal Root. It is to be understood in algebra that *the radical sign indicates the principal root.*

Thus $\sqrt{4}=2$, and it is incorrect to write $\sqrt{4}=-2$, although 4 has two square roots. If we wish to indicate the negative root, we write $-\sqrt{4}=-2$, and if we wish to indicate both roots, we write $\pm\sqrt{4}=\pm 2$.

Therefore, if we have to simplify $\sqrt{4}+\sqrt{9}+\sqrt{16}+\sqrt{25}$, we have $2+3+4+5=14$, and not $\pm 2 \pm 3 \pm 4 \pm 5$, which has several values.

16. Binomial Theorem. By actual multiplication we have

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

In these results it is seen that

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *In the first term the exponent of a is the same as the exponent of the power to which the binomial is raised, and decreases by one in each succeeding term.*

3. *The letter b appears in the second term, with an exponent 1, and the exponent increases by one in each succeeding term.*

4. *The coefficient of the first term is 1.*

5. *The coefficient of each term after the first may be found from the preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

The above law is called the *Binomial Theorem.*

In general,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3} + \cdots$$

If b is negative, the terms in which the *odd* powers of b occur are *negative.* Thus $(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3$, where the terms are alternately positive and negative.

The most important part of the Binomial Theorem is paragraph 5, as stated above, and this should be mastered. The theorem is true for any positive exponent, which is the kind that the student will meet in commercial algebra. It is necessary in understanding square root and in extracting square root on a calculating machine. It is also of value in the theory of compound interest. The proof of this theorem is given in higher algebra and may be assumed by the students who read this book.

17. Illustrative Examples. **1.** Square the binomial $2x - 3$.

Since $$(a-b)^2 = a^2 - 2ab + b^2,$$
we have $$(2x-3)^2 = (2x)^2 - 2\cdot(2x)\cdot 3 + 3^2$$
$$= 4x^2 - 12x + 9.$$

Check. Let $x = 1$.

Then $$(-1)^2 = 1 = 4 - 12 + 9.$$

2. Raise $\frac{3}{5}x^2y^3 - \frac{2}{3}$ to the third power; that is, expand $(\frac{3}{5}x^2y^3 - \frac{2}{3})^3$.

Since $$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$
we have
$$(\tfrac{3}{5}x^2y^3 - \tfrac{2}{3})^3 = (\tfrac{3}{5}x^2y^3)^3 - 3\cdot(\tfrac{3}{5}x^2y^3)^2\cdot\tfrac{2}{3} + 3\cdot(\tfrac{3}{5}x^2y^3)\cdot(\tfrac{2}{3})^2 - (\tfrac{2}{3})^3$$
$$= \tfrac{27}{125}x^6y^9 - \tfrac{18}{25}x^4y^6 + \tfrac{4}{5}x^2y^3 - \tfrac{8}{27}.$$

3. Raise $2x - y^2$ to the fifth power.

Since $$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5,$$
we have $$(2x - y^2)^5 = (2x)^5 - 5(2x)^4(y^2) + 10(2x)^3(y^2)^2$$
$$- 10(2x)^2(y^2)^3 + 5(2x)(y^2)^4 - (y^2)^5$$
$$= 32x^5 - 80x^4y^2 + 80x^3y^4 - 40x^2y^6 + 10xy^8 - y^{10}.$$

Exercise 3. Powers of Binomials

Examples 1 to 7, oral

Expand the following:

1. $(x+y)^2$.
2. $(x+y)^4$.
3. $(x-y)^5$.
4. $(m+n)^6$.
5. $(1+y)^2$.
6. $(1+i)^2$.
7. $(1+r)^3$.
8. $(1+\frac{1}{2}i)^4$.
9. $(3x+5y)^2$.
10. $(2a^2-4b)^2$.
11. $(3a^2+2)^4$.
12. $(x+0.5)^2$.
13. $(x+0.4)^3$.
14. $(3+2r)^4$.
15. $(1+3i)^3$.
16. $(1+i)^5$.
17. $(1+i)^6$.
18. $(1+0.02)^2$.
19. $(1+0.04)^2$.
20. $(1+0.03)^3$.
21. $(1+0.02\frac{1}{2})^2$.
22. $(1+0.05)^2$.
23. $(1+0.06)^2$.
24. $(1+0.03)^4$.

18. Quarter Squares. Since commercial computation demands both rapidity and accuracy, various methods have been developed for the several operations with numbers. These include logarithms and the slide rule (to be explained in the next two chapters), calculating machines of various kinds, and an ingenious device known as *quarter squares.* This last method depends upon a simple application of the Binomial Theorem. Although it is not frequently used in commercial work, logarithms being in general more convenient and practical, it is an interesting device.

Consider first the following relations:

$$\tfrac{1}{4}(a+b)^2 = \tfrac{1}{4}a^2 + \tfrac{1}{2}ab + \tfrac{1}{4}b^2,$$

and

$$\tfrac{1}{4}(a-b)^2 = \tfrac{1}{4}a^2 - \tfrac{1}{2}ab + \tfrac{1}{4}b^2;$$

whence $\tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2 = ab.$

That is, by simple subtraction we can find the product of any two numbers represented by a and b if we only know the numbers represented by $\tfrac{1}{4}(a+b)^2$ and $\tfrac{1}{4}(a-b)^2$; that is, if we know the quarter squares of the numbers represented by $a+b$ and $a-b$.

Hence, if we have a table giving the quarter squares of the sum and the difference of the various numbers that we may care to multiply, we can find the product of the numbers by a single subtraction, thus providing a more simple operation, lessening the chance of error, and saving much time.

Of course $\tfrac{1}{4}$ of a number may have a fractional part, so that we have to consider whether this fact invalidates the use of quarter squares. If $a+b$ is even, so is $a-b$, and the square of an even number, say of $2n$, contains the factor 4, so there is no fraction in that case. If $a+b$ is odd, so is $a-b$, and each is of the form $2n+1$. If the square of $2n+1$, which is $4n^2+4n+1$, is divided by 4, there will be a fraction $\tfrac{1}{4}$ in each quarter square, but these fractions will disappear in the subtraction. Hence the fractions will not trouble us in either case and may be neglected.

19. Table of Quarter Squares. A table of quarter squares is too elaborate to be reproduced on a small page. A single example may be given, however, to illustrate its use.

Required the product of 93,319 and 33,674.

We have

$$\begin{aligned} a &= 93{,}319 \\ b &= 33{,}674 \\ \hline a+b &= 126{,}993 \\ a-b &= 59{,}645 \end{aligned}$$

By a table of quarter squares,

$$\begin{aligned} \tfrac{1}{4} \text{ of } 126{,}993^2 &= 4{,}031{,}805{,}512 \\ \tfrac{1}{4} \text{ of } 59{,}645^2 &= 889{,}381{,}506 \\ \hline \end{aligned}$$

Subtracting, $ab = 3{,}142{,}424{,}006$

We see that the fractions $\frac{1}{4}$ cancel out, as stated on page 9.

Exercise 4. Quarter Squares

1. In multiplying 24,768 by 10,234 a table of quarter squares gives $\frac{1}{4}(a+b)^2$ as 306,285,001 and $\frac{1}{4}(a-b)^2$ as 52,809,289. Find the product, and check the work by means of actual multiplication.

2. What is the quarter square of $36+24$? of $36-24$? Multiply 36 by 24 by means of quarter squares.

3. Multiply 48 by 32 by means of quarter squares, making out your own table for that purpose.

First computing the necessary quarter squares, find the products indicated below, checking by actual multiplication:

4. 12 × 64.
5. 16 × 80.
6. 24 × 40.
7. 32 × 62.
8. 32 × 68.
9. 36 × 88.
10. 16 × 64.
11. 16 × 88.
12. 24 × 44.
13. 24 × 48.
14. 24 × 64.
15. 28 × 32.
16. 21 × 25.
17. 13 × 27.
18. 15 × 49.
19. 23 × 75.
20. 37 × 65.
21. 39 × 73.

20. Square Root of a Polynomial. If we square $a+b$, the result is $a^2+2ab+b^2$. In extracting the square root of $a^2+2ab+b^2$ we reverse the process of squaring, thus:

$$\begin{array}{lll|l}
\text{The given square} = & & a^2+2ab+b^2 & \underline{a+b} = \text{Root} \\
\text{The square of } a = & & a^2 & \\
\cline{2-3}
\text{Trial divisor} = & 2a & \multicolumn{1}{|l}{2ab+b^2} = \text{First remainder} & \\
\text{Complete divisor} = & 2a+b & \multicolumn{1}{|l}{\underline{2ab+b^2}} = b(2a+b) & \\
\end{array}$$

The first term of the root is evidently a, because the square of a is a^2.

Since in squaring a binomial we have the square of the first term plus twice the product of the first and second terms, and so on, we have in $2ab$ twice the product of a and the second term. We therefore divide by $2a$ to find the second term. Since $2ab \div 2a = b$, the second term is b.

In squaring $a+b$ we have $a^2+b(2a+b)$. We therefore add b to $2a$, and multiply $2a+b$ by b, thus completing the square of $a+b$.

The names *trial divisor* and *complete divisor* are used as above shown, being convenient names brought into algebra from arithmetic.

Find the square root of $9x^2-30xy^2+25y^4$.

$$\begin{array}{lll|l}
\text{The given square} = & & 9x^2-30xy^2+25y^4 & \underline{3x-5y^2} = \text{Root} \\
\text{The square of } 3x = & & 9x^2 & \\
\cline{2-3}
\text{Trial divisor} = & 6x & \multicolumn{1}{|l}{-30xy^2+25y^4} & \\
\text{Complete divisor} = & 6x-5y^2 & \multicolumn{1}{|l}{\underline{-30xy^2+25y^4}} & \\
\end{array}$$

The student should compare this example, step by step, with the one explained above, answering the following questions:

How is the first term of the root found? Why is $6x$ taken as the trial divisor? How is the second term of the root found? Why is $-5y^2$ added to $6x$? Why is $6x-5y^2$ multiplied by $-5y^2$?

21. Practical Work in Square Root. The square root of a small number is practically found by the aid of tables of roots or tables of logarithms. For larger numbers, in engineering, a calculating machine is used, but this necessitates a knowledge of square root such as is set forth on this page.

22. Square Root of a Polynomial containing Fractions. If a polynomial contains powers and reciprocals of the same letter, the arrangement in descending powers is as follows:

$$\cdots x^4,\ x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \frac{1}{x^4},\ \cdots$$

Find the square root of $x^2 + 13 + 6x + \frac{4}{x^2} + \frac{12}{x}$.

First arranging according to the descending powers of x, we have

$$
\begin{array}{l|l|l}
 & x^2 + 6x + 13 + \frac{12}{x} + \frac{4}{x^2} & x + 3 + \frac{2}{x} \\
 & x^2 & \\ \hline
2x & 6x + 13 & \\
2x + 3 & 6x + 9 & \\ \hline
2x + 6 & 4 + \frac{12}{x} + \frac{4}{x^2} & \\
2x + 6 + \frac{2}{x} & 4 + \frac{12}{x} + \frac{4}{x^2} & \\ \hline
\end{array}
$$

Check. Let $x = 2$, so as to avoid fractions. We then have

$$4 + 13 + 12 + 1 + 6 = 36, \text{ and } 2 + 3 + 1 = 6 = \sqrt{36}.$$

23. Extracting the Square Root. Therefore, in extracting the square root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the principal square root of the first term, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.

Multiply the sum of twice the first term and the second term of the root by the second term, thus completing the square of the sum of the first two terms, and subtract this product.

Consider that part of the root which has now been found as the first term of the root, and proceed as before.

Proceed in this manner until all the terms of the given polynomial have been used.

Exercise 5. Square Roots of Polynomials

Examples 1 and 2, oral

1. State the square roots of $a^2+2+\frac{1}{a^2}$; $9a^2+6+\frac{1}{a^2}$.

2. State the square roots of $x^2-8+\frac{16}{x^2}$; $\frac{1}{a^2}-\frac{6}{ab}+\frac{9}{b^2}$.

Find the square roots of the following expressions:

3. $\frac{a^2}{9}+\frac{2a}{3}+\frac{7}{3}+\frac{4}{a}+\frac{4}{a^2}$.

4. $\frac{4}{b^2}+25a^4+\frac{9}{c^4}+\frac{20a^2}{b}-\frac{30a^2}{c^2}-\frac{12}{bc^2}$.

5. $9+\frac{4b^2}{25a^4}+\frac{2}{5ab}-\frac{12b}{5a^2}-\frac{3a}{b^2}+\frac{a^2}{4b^4}$.

6. $\frac{4r^2}{a^2}-\frac{4pr}{aq}+\frac{p^2}{q^2}+\frac{6sp}{bq}+\frac{9s^2}{b^2}-\frac{12sr}{ab}$.

7. $\frac{4n^{30}}{a^{18}}-\frac{2m^3n^{12}}{3a^3}+\frac{n^{18}}{9a^6}+\frac{4m^3n^{18}}{a^9}-\frac{4n^{24}}{3a^{12}}+m^6n^6$.

8. $\frac{9x^2}{y^2}+y^2+\frac{y^2}{9x^2}+6x+2+\frac{2y^2}{3x}$.

9. $\frac{1}{16}r^8+\frac{1}{9}q^6+\frac{1}{4}a^2r^4+\frac{1}{4}a^4-\frac{1}{3}a^2q^3-\frac{1}{6}q^3r^4$.

10. If the square root of $a^4+4a^2+9+4a^3-6a^2-12a$ is a^2+xa-y, what are the values of x and y?

11. If the square root of $a^2+xa+25$ is $a+y$, what are the values of x and y?

12. It is known that $x^4+8x^3+30x^2+56x+49$ is an exact square of a trinomial. Without going through the complete process of extracting the square root, write the first term of the root; the last term of the root; the second term of the root; the entire root. Check the result.

24. Arithmetic Square Root. The first step in extracting the square root of a number is to separate the number into groups of two figures each, called *periods.*

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10, and the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* figure or *two* figures is a number of *one* figure; the square root of any integral number expressed by *three* figures or *four* figures is a number of *two* figures; and so on.

1. Find the square root of 2209.

Let t = tens and u = units, and observe that $(t + u)^2 = t^2 + 2\,tu + u^2$.

Separating the given number into periods, we see that there will be two integral places in the root.

The first period, 22, contains t^2. Since the greatest square in 22 is 16, it follows that 4, or $\sqrt{16}$, is the tens' figure of the root, or t.

Subtracting t^2, the remainder is $2\,tu + u^2$. Therefore, if we divide by $2\,t$ (that is, by 80, which is 2×4 tens), we shall find approximately u, the units. Dividing 609 by 80 (or 60 by 8), we have 7 as the units' figure.

$$\begin{array}{ll}
 & 22\ 09(47 \\
 & 16 \\ \hline
80 & 6\ 09 \\
87 & 6\ 09 \\ \hline
\end{array}$$

Since $2\,tu + u^2 = (2\,t + u)\,u$, that is, $2 \times 40 \times 7 + 7^2 = (2 \times 40 + 7) \times 7$, we add 7 to 80 and multiply the sum by 7. The product is 609, thus completing the square of 47.

2. Find the square root of 7,890,481.

When the third period, 04, is brought down, the next figure of the root is 0, because 560, the trial divisor, is not contained in 504. Hence 0 is placed in the root, another 0 is annexed to the divisor, and the next two figures, 81, are brought down.

$$\begin{array}{ll}
 & 7\ 89\ 04\ 81(2809 \\
 & 4 \\ \hline
40 & 3\ 89 \\
48 & 3\ 84 \\ \hline
5600 & \quad 5\ 04\ 81 \\
5609 & \quad 5\ 04\ 81 \\ \hline
\end{array}$$

The work then proceeds as before, the trial divisor now being 5600, which is twice 280 tens. The entire root is thus found to be 2809.

25. Approximate Square Roots. If a number is not a perfect square, zeros may be annexed and an approximate value of the root found, as in the case of $\sqrt{19}$.

In this example we proceed in the usual way, annexing a pair of zeros for each decimal place in the root, and separating into periods of two figures each to the right and left of the decimal point. We carry the work to four decimal places if we wish to find the nearest approximation for three places. The result is found to be 4.359; that is, it is nearer 4.359 than 4.358.

	19.00 00 00 (4.3588 +
	16
80	3 00
83	2 49
860	51 00
865	43 25
8700	7 75 00
8708	6 96 64
87160	78 36 00

26. Summary of Square Root. We may summarize thus:

Separate the number into periods of two figures each, beginning at the decimal point and working toward the left for the integer and toward the right for the decimal.

Find the greatest square in the first period at the extreme left and write its root for the first figure of the required root.

Square this root, subtract the result from the period at the left, and to the remainder annex the next period for a dividend.

For a partial divisor double the root already found, considered as tens, and divide the dividend by it. The quotient (diminished slightly if necessary) is the next figure of the root.

To the partial divisor annex this next figure of the root for a complete divisor. Multiply the complete divisor by this next figure of the root, subtract the product from the dividend, and to the remainder annex the next period for a new dividend.

Proceed in this manner until all the periods have been thus annexed. The result is the square root required.

27. Applications of Square Root. The following statements are proved in geometry:

The corresponding lines of similar figures are to each other as the square roots of the areas of the figures.

That is, if one circle has 16 times the area of another, the radius is 4 times as long as that of the other.

The radius of a circle is approximately equal to the square root of the quotient of the area of the circle divided by 3.1416.

For example, if the area of a circle is 78.54 sq. in., the number of inches in the radius is $\sqrt{\frac{78.54}{3.1416}} = \sqrt{25} = 5$. Therefore the radius is 5 in.

In the following exercise carry approximate square roots to the nearest hundredth. In Exs. 16–20 first reduce each fraction to a decimal. In Exs. 5, 10, and 15 extract the square root of each term separately.

Exercise 6. Square Root

Find the square root of each of the following:

1. 12,996.
2. 164.8656.
3. 192.6544.
4. 3863.8656.
5. $\frac{400}{441}$.
6. 2,371,600.
7. 627,264.
8. 1,707,552.
9. 12,446,784.
10. $\frac{676}{961}$.
11. 8.
12. 39.
13. 515.
14. 18.17999044.
15. $\frac{1156}{1369}$.
16. $\frac{1}{6}$.
17. $\frac{1}{7}$.
18. $\frac{1}{10}$.
19. $\frac{5}{8}$.
20. $\frac{7}{8}$.

Find the radii of circles whose areas are:

21. 2513.28 sq. in.
22. 452.3904 sq. in.
23. 700 sq. in.

24. What must be the diameter of a water main in order that the area of the cross section may be 4 sq. ft.?

25. A tinsmith wishes to make a cylindric gallon can 8 in. high. What must be the area of the base? What radius must he use to draw the base? (1 gal. = 231 cu. in.)

28. Fractional Exponent. A positive integral exponent of an expression shows how many times the expression which it affects is taken as a factor.

Thus $x^2 = xx$, the exponent showing that x is taken twice as a factor.

In our study of logarithms we shall need fractional exponents, but these must have a different meaning from integral exponents, for we cannot take x half of a time as a factor any more than we can pick up a book half of a time.

Since we have certain laws of positive intégral exponents, it would be illogical and inconvenient if we should give to fractional exponents a meaning such that these laws would not also apply to them. Therefore

Such a meaning must be given to fractional exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \cdot a^m = a^{m+m} = a^{2m}$,

we must have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

But $\sqrt{a} \cdot \sqrt{a} = a$.

We therefore define $a^{\frac{1}{2}}$ to mean $\sqrt{a}$.

Since $(a^m)^n = a^{mn}$,

we must have $(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}$.

We therefore define $a^{\frac{3}{2}}$ to mean $(\sqrt{a})^3$.

In a fractional exponent the numerator indicates a power of the expression affected, and the denominator indicates a root.

Thus $4^{\frac{1}{2}}$ means the same as $\sqrt{4}$, and hence $4^{\frac{1}{2}} = 2$;

$8^{\frac{2}{3}}$ means the same as $(\sqrt[3]{8})^2$, or as $\sqrt[3]{8^2}$, and hence $8^{\frac{2}{3}} = 4$,

and $a^{\frac{m}{n}}$ means the same as $(\sqrt[n]{a})^m$, or as $\sqrt[n]{a^m}$.

In a case like that of $8^{\frac{2}{3}}$ it is easier to extract the root first and then to raise to the power, but the opposite order is better in a case like that of $7^{\frac{2}{3}}$, because we can find $\sqrt[3]{7}$ only approximately.

29. Comparison of Fractional Exponents and Radicals. The following comparison of fractional exponents with the ordinary radical sign should be studied with care:

Fractional Exponent	*Ordinary Radical Sign*
$(a^{\frac{1}{n}})^n = a$	$(\sqrt[n]{a})^n = a$
$(a^n)^{\frac{1}{n}} = a$	$\sqrt[n]{a^n} = a$
$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
$(a^{mn})^{\frac{1}{n}} = a^m$	$\sqrt[n]{a^{mn}} = a^m$
$ab^{\frac{1}{n}} = (a^nb)^{\frac{1}{n}}$	$a\sqrt[n]{b} = \sqrt[n]{a^nb}$
$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
$(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}$	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

For example, $(x^{\frac{1}{3}})^3 = x$, and $(\sqrt[3]{x})^3 = x$;

$(y^4)^{\frac{1}{4}} = y$, and $\sqrt[4]{y^4} = y$;

$(z^6)^{\frac{1}{2}} = z^3$, and $\sqrt{z^6} = z^3$;

$$\left(\frac{9}{16}\right)^{\frac{1}{2}} = \frac{9^{\frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{3}{4}, \text{ and } \sqrt{\frac{9}{16}} = \frac{3}{4};$$

$$(1+i)^{\frac{3}{2}}(1+i)^{\frac{1}{2}} = (1+i)^{\frac{4}{2}} = (1+i)^2.$$

30. Reduction of Fractional Exponents. Since the nth root of any power of a number is found by dividing the exponent of the power by the index of the root, in the expression $a^{\frac{m}{n}}$ we may divide m by n just as if the exponent $\frac{m}{n}$ were an ordinary fraction. That is,

A fractional exponent may be reduced in the same way as an ordinary fraction.

Thus, $a^{\frac{2}{4}} = a^{\frac{1}{2}}$, $x^{\frac{2}{3}} = x^{\frac{4}{6}}$, $m^{\frac{4}{4}} = m$, $(a+b)^{\frac{3}{9}} = (a+b)^{\frac{1}{3}}$, $(a-b)^{\frac{6}{2}} = (a-b)^3$.

Exercise 7. Fractional Exponents

Examples 1 to 5, oral

1. What is the meaning of $x^{\frac{1}{2}}$? $x^{\frac{1}{4}}$? $x^{\frac{1}{6}}$? $x^{\frac{1}{9}}$?

2. What is the meaning of $a^{\frac{5}{8}}$? $(mn)^{\frac{1}{2}}$? $m^{\frac{1}{2}}n^{\frac{1}{2}}$? $m^{\frac{2}{3}}n^{\frac{3}{8}}$?

3. What is the value of $9^{\frac{1}{2}}$? $64^{\frac{1}{2}}$? $49^{\frac{1}{2}}$? $169^{\frac{1}{2}}$? $900^{\frac{1}{2}}$?

4. What is the value of $16^{\frac{1}{4}}$? $32^{\frac{1}{5}}$? $32^{\frac{3}{5}}$? $8^{\frac{2}{3}}$? $32^{\frac{4}{5}}$?

5. What is the value of $4^{\frac{1}{2}} \cdot 8^{\frac{1}{3}}$? $4^{\frac{1}{2}} \cdot 8^{\frac{2}{3}}$? $9^{\frac{1}{2}} \cdot 8^{\frac{2}{3}}$?

Express with radical signs:

6. $x^{\frac{3}{4}}$.

7. $y^{\frac{3}{5}}$.

8. $z^{\frac{3}{8}}$.

9. $3x^{\frac{1}{4}}$.

10. $(3x)^{\frac{1}{2}}$.

11. $3^{\frac{1}{4}}x^{\frac{1}{4}}$.

12. $9x^{\frac{3}{4}}$.

13. $(9x)^{\frac{3}{4}}$.

14. $9^{\frac{3}{4}}x^{\frac{3}{4}}$.

15. $m^{\frac{1}{3}}n^{\frac{1}{3}}$.

16. $m^{\frac{1}{3}}n^{\frac{1}{3}}p^{\frac{1}{3}}$.

17. $a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}$.

Simplify the following:

18. $49^{\frac{1}{2}}$.

19. $27^{\frac{1}{3}}$.

20. $27^{\frac{2}{3}}$.

21. $125^{\frac{2}{3}}$.

22. $-27^{\frac{1}{3}}$.

23. $-27^{\frac{2}{3}}$.

24. $(-32)^{\frac{1}{5}}$.

25. $(-32)^{\frac{2}{5}}$.

26. $32^{\frac{2}{5}} \cdot 27^{\frac{1}{3}}$.

27. $(\frac{1}{16})^{\frac{1}{2}} \cdot (\frac{1}{36})^{\frac{1}{2}}$.

28. $9^{\frac{1}{2}} \cdot (\frac{1}{9})^{\frac{1}{2}} \cdot 25^{\frac{1}{2}}$.

29. $125^{\frac{2}{3}} \cdot 144^{\frac{1}{2}}$.

Express with fractional exponents:

30. $\sqrt{m}$.

31. $\sqrt{mn}$.

32. $\sqrt[3]{a^2b^2}$.

33. $\sqrt[3]{8ab^2}$.

34. $\sqrt[4]{a^3b^3}$.

35. $\sqrt[5]{a^3b^5}$.

36. $\sqrt[5]{32a^5b^7}$.

37. $\sqrt[3]{m^3}$.

38. $\sqrt[3]{m^3n}$.

39. $\sqrt[5]{a^5b^5}$.

40. $3\sqrt{a}\sqrt[3]{b^2}$.

41. $8\sqrt[3]{a}\sqrt[4]{n^3}$.

42. $3^3 \cdot \sqrt[3]{a^2}\sqrt[4]{a^5}$.

43. $5a^3\sqrt[4]{a}\sqrt[3]{b^2}$.

44. $\sqrt[8]{a}\sqrt[8]{b}$.

45. $5\sqrt[3]{(mn)^3}$.

46. $\sqrt{(a+b)^3}$.

47. $b\sqrt{b} \cdot d^2\sqrt[3]{d}$.

48. $\sqrt[n]{x} \cdot \sqrt[m]{y}$.

49. $\sqrt[n]{x^m} \cdot \sqrt[p]{y^q}$.

50. $\sqrt[n]{x^{2m}} \cdot \sqrt[p]{y^p}$.

31. Negative Exponent. We have found the meaning of positive integral exponents as follows:

a^2 means aa,

a^3 means aaa,

and, in general, a^n means $aaa \cdots$ to n factors.

We have also found the meaning of fractional exponents, as follows:

$a^{\frac{2}{3}}$ means $\sqrt[3]{a^2}$ or $(\sqrt[3]{a})^2$,

$a^{\frac{m}{n}}$ means $\sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$.

In our work with logarithms we shall have occasion to use negative exponents, and their meaning will now be considered. As with fractional exponents,

Such a meaning must be given to negative exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \div a^n = a^{m-n}$, we know that $a^5 \div a^2 = a^3$. We must therefore have $a^2 \div a^5 = a^{2-5} = a^{-3}$. But $a^2 \div a^5 = \frac{1}{a^3}$. We must therefore have $a^{-3} = \frac{1}{a^3}$.

More generally, we must have $a^m \cdot a^{-n} = a^{m-n}$; but $a^m \cdot \frac{1}{a^n}$ is equal to a^{m-n}, and hence a^{-n} must be equal to $\frac{1}{a^n}$.

Therefore, *an expression affected by a negative exponent is equal to the reciprocal of that expression affected by a numerically equal positive exponent.*

That is, $$\boldsymbol{a^{-n} = \frac{1}{a^n}}.$$

Hence $$a^{-2} = \frac{1}{a^2}, \qquad a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}},$$

$$(1+i)^{-3} = \frac{1}{(1+i)^3}, \qquad a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}.$$

32. Zero as an Exponent. It now remains to find the meaning of zero as an exponent. If a^0 is to conform to the same laws as a^m, we must have $a^0 \cdot a^m = a^{0+m} = a^m$. But to have this true, a^0 must equal 1. Therefore, in general,

An expression affected by an exponent zero is equal to 1.

That is, $$a^0 = 1.$$

For example, $2^0 = 1$, $(\frac{1}{2})^0 = 1$, $(-1)^0 = 1$, and $x^0 = 1$.

Exercise 8. Negative and Zero Exponents

Examples 1 to 22, oral

1. Express x^{-3}, using a positive exponent.

2. What is the value of 9^0? of $(-8)^0$? of $(\frac{1}{3})^0$? of a^0? of 5^0? of 5^{-1}? of $9^{-\frac{1}{2}}$? of $4^{-\frac{1}{2}}$? of $8^{-\frac{1}{3}}$?

Express the following with positive exponents:

3. m^{-5}.
4. a^{-7}.
5. $(\frac{1}{3})^{-1}$.
6. $(\frac{1}{4})^{-1}$.
7. $(\frac{1}{4})^{-2}$.
8. $(\frac{1}{2})^{-5}$.
9. $27^{-\frac{1}{3}}$.
10. $27^{-\frac{2}{3}}$.
11. $64^{-\frac{2}{3}}$.
12. $x^{-\frac{2}{3}}$.
13. $(x+y)^{-1}$.
14. $x^{-1}+y^{-1}$.

Express the following in integral form, using negative and fractional exponents:

15. $\frac{1}{3}$.
16. $\dfrac{1}{m}$.
17. $\frac{3}{4}$.
18. $\dfrac{3}{x^3}$.
19. $\frac{1}{36}$.
20. $\dfrac{p}{2\sqrt{q}}$.
21. $\frac{1}{121}$.
22. $\dfrac{p}{q\sqrt{r}}$.

Copy and express the following in integral form, using negative and fractional exponents:

23. $\dfrac{a}{x^2}$.
24. $\dfrac{a}{\sqrt{xy}}$.
25. $\dfrac{a^2+b^2}{a-b}$.
26. $\dfrac{a+b}{\sqrt{a^2-b^2}}$.
27. $\dfrac{1}{a^2+4\,ab+4\,b^2}$.
28. $\dfrac{1}{\sqrt[3]{a^2-6\,ab+9\,b^2}}$.

33. Operations involving Negative, Zero, and Fractional Exponents. Since we have defined negative, zero, and fractional exponents so that they conform to the ordinary laws of positive integral exponents, we operate with them in the same way as with positive integral exponents.

In the following operations the quantities having negative exponents are compared with the fractional forms to show their relation. In solving the problems in Exercise 9, use negative exponents whenever they appear, without changing to fractional forms unless so directed.

1. Multiply $x^{-1}+2$ by $x^{-1}-7$.

$$\begin{aligned}&(x^{-1}+2)(x^{-1}-7)\\&\quad=x^{-2}+2x^{-1}-7x^{-1}-14\\&\quad=x^{-2}-5x^{-1}-14.\end{aligned}$$

It is seen that the quantities with the negative exponents occupy less space and are quite as easily written.

$$\begin{aligned}&\left(\frac{1}{x}+2\right)\left(\frac{1}{x}-7\right)\\&=\frac{1}{x^2}+\frac{2}{x}-\frac{7}{x}-14\\&=\frac{1}{x^2}-\frac{5}{x}-14.\end{aligned}$$

2. Divide $x^{-3}+1$ by $x^{-1}+1$.

$$\begin{array}{l|l}
x^{-3}+1 & x^{-1}+1\\ \hline
x^{-3}+x^{-2} & x^{-2}-x^{-1}+1\\ \hline
\quad -x^{-2}+1 & \\
\quad -x^{-2}-x^{-1} & \\ \cline{1-1}
\qquad\qquad x^{-1}+1 & \\
\qquad\qquad x^{-1}+1 & \\ \cline{1-1}
\end{array}$$

This work is more compact in form than the other and is more easily written. After a little experience this form is understood as easily as the other.

$$\begin{array}{l|l}
\frac{1}{x^3}+1 & \frac{1}{x}+1\\ \hline
\frac{1}{x^3}+\frac{1}{x^2} & \frac{1}{x^2}-\frac{1}{x}+1\\ \hline
\quad -\frac{1}{x^2}+1 & \\
\quad -\frac{1}{x^2}-\frac{1}{x} & \\ \cline{1-1}
\qquad\qquad \frac{1}{x}+1 & \\
\qquad\qquad \frac{1}{x}+1 & \\ \cline{1-1}
\end{array}$$

3. Factor $x^{-2}-25$.

Since we have the difference of two squares, we may factor as in the case of a^2-b^2, writing fractions in the result if we wish.

Hence $$x^{-2}-25=(x^{-1}+5)(x^{-1}-5).$$

34. Advantage of Negative and Fractional Exponents. The advantage of using a^{-n} instead of $1/a^n$, and $a^{\frac{3}{4}}$ instead of $\sqrt[4]{a^3}$ should be fully appreciated by the student.

Thus it is easier to see that $(a^{-\frac{3}{4}})^{-\frac{4}{3}} = a$, than that

$$\frac{1}{\sqrt[3]{\left(\frac{1}{\sqrt[4]{a^3}}\right)^4}} = a,$$

although the two mean the same thing. In our work with logarithms we shall frequently need to use zero, negative, and fractional exponents.

In scientific work it is quite customary to write $4 \cdot 10^{-7}$ instead of $\frac{4}{10000000}$ or 0.0000004 and $5 \cdot 10^{15}$ instead of 5,000,000,000,000,000.

Exercise 9. Negative and Fractional Exponents

Examples 1 to 5, oral

1. Simplify $(x^{-2})^{-5}$; $(x^{\frac{1}{2}})^3$; $(x^{\frac{2}{3}})^3$; $(y^{\frac{2}{3}})^{\frac{3}{2}}$.
2. Expand $(x^{-1} + y^{-1})^2$; $(x^{-2} - y^{-2})^2$; $(x^{-2} + y^2)^2$.
3. Multiply $x^{-1} + y^{-1}$ by $x^{-1} - y^{-1}$; by $x^{-1} + y^{-1}$.
4. Divide $x^{-2} - y^{-2}$ by $x^{-1} + y^{-1}$; by $x^{-1} - y^{-1}$.
5. Expand $(p^{\frac{1}{2}} + q^{\frac{1}{2}})^2$; $(p^{\frac{1}{2}} - q^{\frac{1}{2}})^2$; $(p^{\frac{1}{2}} + q^{-1})^2$.

Perform the operations indicated:

6. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.
7. $(a^2 + a^{-2})(a^2 - a^{-2})$.
8. $(x^{\frac{1}{m}} + y^{\frac{1}{m}})^2$.
9. $(p^{-\frac{1}{2}} + q^{-\frac{1}{2}})(p^{-\frac{1}{2}} - q^{-\frac{1}{2}})$.
10. $(p^2q^2 + 3\,q^{-2})(p^2q^2 - 3\,q^{-2})$.
11. $(x^{-\frac{p}{q}} + y^{\frac{p}{q}})(x^{-\frac{p}{q}} - y^{\frac{p}{q}})$.
12. Divide $a^{-10} - a^{-5} + 1$ by $a^{-2} - a^{-1} + 1$.
13. Divide $m^{-3} + 2\,m^{-2}n^{-1} - 3\,n^{-3}$ by $m^{-1} - n^{-1}$.
14. Expand $(1 + v^{-1})^3$, using only positive exponents.
15. Expand $(1 + i^{-2})^4$, using only positive exponents.

Perform the following multiplications:

16. $a^{\frac{1}{2}} + 2a^{\frac{1}{4}} + 1$ by $a^{\frac{1}{2}} + 1$.

17. $a^{\frac{3}{4}} + a^{\frac{1}{2}} + a^{\frac{1}{4}} + 1$ by $a^{\frac{1}{2}} + a^{\frac{1}{4}} + 1$.

18. $m^{\frac{3}{4}} + 2m^{\frac{3}{2}} + 3m$ by $m^{\frac{3}{4}} - 2m^{\frac{3}{2}} + 3m$.

19. $x + 6a^{\frac{1}{3}}x^{\frac{2}{3}} + 12a^{\frac{2}{3}}x^{\frac{1}{3}} + 8a$ by $x^{\frac{2}{3}} + 4a^{\frac{1}{3}}x^{\frac{1}{3}} + 4a^{\frac{2}{3}}$.

20. $a^{-3} - a^{-2} + a^{-1} - 1$ by $a^{-2} - a^{-1} + 1$.

21. $a^{-\frac{2}{3}} + 2a^{-\frac{1}{3}}b^{-\frac{1}{3}} + b^{-\frac{2}{3}}$ by $a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}}b^{-\frac{1}{3}} + b^{-\frac{2}{3}}$.

22. $m^{-\frac{1}{2}} - 2m^{-\frac{1}{4}}n^{-\frac{1}{3}} + n^{-\frac{2}{3}}$ by $m^{-\frac{1}{2}} + 2m^{-\frac{1}{4}}n^{-\frac{1}{3}} + n^{-\frac{2}{3}}$.

Perform the following divisions:

23. $prx^2y^{-1} - qryx^{-1}$ by $rx^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

24. $a^2 + 3a^{\frac{5}{3}} + 6a^{\frac{4}{3}} + 7a + 6a^{\frac{2}{3}} + 3a^{\frac{1}{3}} + 1$ by $a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1$.

25. $\frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}} - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab + \frac{9}{16}a^2$ by $\frac{1}{3}b^{\frac{1}{2}} + \frac{3}{2}a^{\frac{1}{2}}$.

26. $x^{-\frac{7}{3}} - 3x^{-2} + x^{-\frac{5}{3}} - 4 + 12x^{-\frac{1}{3}} - 4x^{-\frac{2}{3}}$ by $x^{-\frac{5}{3}} - 4$.

27. $x^{-\frac{4}{3}} + x^{-\frac{2}{3}}z^{-\frac{2}{3}} - y^{-\frac{4}{3}} + y^{-\frac{2}{3}}z^{-\frac{2}{3}}$ by $x^{-\frac{2}{3}} + y^{-\frac{2}{3}}$.

Find the square root of each of the following:

28. $p^{\frac{4}{3}}q^{-1} + 4pq^{-\frac{1}{2}} - 2p^{\frac{2}{3}} - 12p^{\frac{1}{3}}q^{\frac{1}{2}} + 9q$.

29. $16a^2b^2 + 25a^{\frac{2}{3}}b + 60ab^{\frac{5}{4}} - 4a^{\frac{4}{3}}b^{\frac{3}{2}} - 48a^{\frac{5}{3}}b^{\frac{7}{4}}$.

30. $25b^{-\frac{4}{3}} + 24a^{-1}b^{-\frac{1}{3}} + 46a^{-\frac{2}{3}}b^{-\frac{2}{3}} + 40a^{-\frac{1}{3}}b^{-1} + 9a^{-\frac{4}{3}}$.

31. Express $9 \cdot 10^{-9}$ as a decimal fraction.

32. Simplify the expression $(1+i)^2 \cdot (1+i)^3 \cdot (1+i)^{-5}$.

33. Write the expression

$$\frac{3a^2}{2b} + \frac{5a}{b^{-2}} - \frac{6c^2d}{13a^{-2}}$$

in the form of an integral polynomial, using negative exponents when necessary.

Exercise 10. Review of Book I

State the meaning of each of the following formulas:

1. $i = prt$.
2. $a = \pi r^2$.
3. $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$.
4. $a = p + i$.
5. $a = p + prt$.
6. $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$.

7. Prove that the excess of 9's in a product is equal to the excess in the product of the excesses of the factors.

Take two factors, $9n + e$ and $9n' + e'$. Why is this legitimate?

8. Using the formula $a(b - c) = ab - ac$, find the product of 9996 and 835,678.

9. A buys 500 yd. of cloth at a gross cost of $1.45 per yard and sells it at a profit of 20% on the selling price, the cost of doing business being covered by 16% of the selling price. Find the selling price and the cost of doing business to be apportioned to the whole transaction.

10. Given $A = p + prt$, derive formulas for p, r, and t.

11. In Ex. 10, given that A = amount, p = principal, r = rate of interest, and t = time expressed in years, make and solve three problems to illustrate the three formulas derived.

12. From the formula $i = prt$ derive a banker's one-day rule for finding interest at $2\frac{1}{4}\%$ and apply this rule to finding the bank discount on $9500 for 40 da. at $2\frac{1}{4}\%$.

13. Given that s is the total receipts from sales, n the number of articles sold, and c the cost of the n articles, all values being expressed in dollars, plot on the same axes the equations $s = 2.5n$ and $c = 500 + 1.5n$. From the graph determine the profit on the sale of 1000 articles and find the number that must be sold to "break even."

Exercise 11. Review of Chapter I

Express each of the following with a single exponent:

1. $(b^3)^4$. **2.** $[(1+i)^5]^2$. **3.** $(1+r)^2(1+r)^3$.

4. Change the form of the expression

$$A=\frac{R}{i}\left[\frac{(1+i)^n-1}{(1+i)^n}\right]$$

by dividing by $(1+i)^n$ both terms of the fraction which is inclosed in the brackets.

5. By the Binomial Theorem expand 1.04^7 to five terms, evaluate the result, and thus find the value of 1.04^7 correct to four decimal places.

6. By the aid of the Binomial Theorem find the value of 1.02^9 correct to two decimal places.

Find the square root of each of the following:

7. 582,169. **8.** 400,725. **9.** 0.0175. **10.** 7.06.

11. $9x^6-26x^4+12x^5+4x^3+16-40x+41x^2$.

In Exs. 7–10 carry the results to three decimal places in the cases of imperfect squares.

12. If the dimensions of a rectangular field have the ratio $3:4$ and the area is 588 sq. rd., what are the dimensions?

13. Find to two decimal places the value of $10^{\frac{3}{2}}$.

14. Arrange $4^{\frac{3}{2}}$, $34^{\frac{1}{2}}$, and 3^2 in order of magnitude, beginning with the smallest.

15. Find the value of $4^{\frac{1}{2}}+25^0-8^{\frac{1}{3}}+\frac{1}{2}-4^{-\frac{1}{2}}+0^4-1^7$.

16. Multiply $x^{-2}+5x^{-1}-7$ by $3x^{-1}+2$.

17. Express the fraction $3x^{-2}y^5/z^{-1}s^2t^{-4}$ as a fraction involving only positive exponents.

CHAPTER II

LOGARITHMS

35. Importance of Logarithms. In many kinds of commercial and scientific work it becomes necessary to rapidly multiply or divide large numbers, to raise numbers to powers, and to find roots of numbers, and all this is very conveniently done by means of logarithms. The use of these aids to computation is easily learned, the theory is important in the study of investments, and before continuing in commercial algebra the student should become familiar with the subject.

36. Logarithm. The exponent of the power to which a given number, called the *base*, must be raised to equal another number is called the *logarithm* of this other number.

For example, since $10^3 = 1000$,

therefore, to the base 10, $3 =$ the logarithm of 1000.

In general, if $b^x = N$,

then, to the base b, $x =$ the logarithm of N.

37. Symbolism. For "logarithm of N" it is customary to write $\log N$ and to read it simply "log N." If we wish to specify $\log N$ to the base b, we write $\log_b N$, reading this, "log N to the base b." Unless the base is stated, 10 is understood to be the base used.

It is evident from § 36 that, to the base 10, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, $\log 10{,}000 = 4$, and so on.

Expressed as a *logarithmic equation*, $10^2 = 100$ becomes $2 = \log 100$ or $\log 100 = 2$.

Expressed as an *exponential equation*, $\log 100 = 2$ becomes $100 = 10^2$.

38. Base. We may take various bases for systems of logarithms, but for practical calculation 10 is taken.

Thus, since $2^3 = 8$, therefore $\log_2 8 = 3$.

If we take 10 as the base, since $10^2 = 100$, and $10^3 = 1000$, we know that the logarithm of any number between 100 and 1000 must lie between 2 and 3. Hence, for example, we know that log 475 is 2 + some fraction.

By using 10 as the base, therefore, we can tell at once the integral part of the logarithm of any number.

39. Logarithm of the Base, of One, and of a Fraction. If we consider the values of 10^1, 10^0, and 10^{-1}, we see that

because $10^1 = 10$, it follows that $\log 10 = 1$;

because $10^0 = 1$, it follows that $\log 1 = 0$;

and because $10^{-1} = \frac{1}{10}$, it follows that $\log \frac{1}{10} = -1$.

That is, *the logarithm of 10 is 1, the logarithm of 1 is 0, and the logarithm of a proper fraction is negative.*

Exercise 12. Logarithms

1. Since $3^2 = 9$, what is $\log_3 9$?
2. Since $10^4 = 10{,}000$, what is $\log 10{,}000$?

Write the following logarithms:

3. $\log_2 16$.
4. $\log_2 64$.
5. $\log_3 27$.
6. $\log_4 16$.
7. $\log_6 36$.
8. $\log_7 49$.
9. $\log 100{,}000$.
10. $\log 1{,}000{,}000$.
11. What is $\log \frac{1}{100}$, or $\log 0.01$? $\log 0.001$? $\log 0.0001$?
12. Between what integers is $\log 425$? $\log 62{,}545$?
13. Between what negative integers is $\log 0.05$? $\log 0.009$?
14. Write the logarithm of $\frac{1}{10000}$ to the base 10.

Write the integers between which the following logarithms lie:

15. $\log 93$.
16. $\log 827$.
17. $\log 5237$.
18. $\log 82{,}759$.

40. Logarithm of a Product. *The logarithm of the product of two factors is equal to the sum of the logarithms of the factors.*

Let A and B be the factors, and x and y their logarithms. Then, taking 10 as the base, and remembering that $x = \log A$ and $y = \log B$, we have

$$A = 10^x,$$

and $$B = 10^y.$$

Multiplying, we have $$AB = 10^{x+y},$$

and therefore $$\log AB = x + y.$$

That is, $$\mathbf{\log} \boldsymbol{AB} = \mathbf{\log} \boldsymbol{A} + \mathbf{\log} \boldsymbol{B}.$$

The proof is the same if any other base is taken.

For if $A = b^x$, and $B = b^y$, we see that $x = \log_b A$, and $y = \log_b B$. Furthermore, $AB = b^{x+y}$, whence $\log_b AB = x + y = \log_b A + \log_b B$. This proof is the same as the one given above, with 10 replaced by b.

The proposition is evidently true for the product of any number of factors, as in the case of

$$\log ABC = \log A + \log B + \log C.$$

41. Logarithm of a Quotient. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For if $x = \log A$, then $$A = 10^x,$$

and if $y = \log B$, then $$B = 10^y.$$

Dividing, we have $$\frac{A}{B} = 10^{x-y},$$

and therefore $$\log \frac{A}{B} = x - y.$$

That is, $$\mathbf{\log} \frac{\boldsymbol{A}}{\boldsymbol{B}} = \mathbf{\log} \boldsymbol{A} - \mathbf{\log} \boldsymbol{B}.$$

It is therefore seen from §§ 40 and 41 that if we know the logarithms of all numbers, we can find the logarithm of a product by addition and the logarithm of a quotient by subtraction.

Exercise 13. Products and Quotients

1. If we know that log 20.4 is 1.3096 and that log 3.58 is 0.5539, what is the logarithm of the product of 20.4 and 3.58?

We find these logarithms from tables which will be explained later.

2. In Ex. 1, if we know that 1.8635 is the logarithm of 73.03, what does this tell us of the product of 20.4 and 3.58? State the reason for the answer.

3. In Ex. 1, if we know that 0.7557 is the logarithm of 5.698, how can we find the value of 20.4 ÷ 3.58?

4. If we know that log 3.41 is 0.5328, log 26.7 is 1.4265, and log 652 is 2.8142, what is the logarithm of the product of 3.41, 26.7, and 652?

5. In Ex. 4, if we know that 4.7735 is the logarithm of 59,400, what is the product of 3.41, 26.7, and 652?

This gives the product to three significant figures only, the two zeros at the right having no significance except to show that the number is approximately 594 hundreds. These details will be explained fully a little later in our work.

6. If we know that log 45.8 is 1.6609 and that log 3.09 is 0.4900, find the logarithm of 3.09 × 45.8 and of 45.8 ÷ 3.09.

7. In Ex. 6, if we know that 2.1509 is the logarithm of 141.5, what does this tell us as to either the product or the quotient mentioned?

8. If you had a table which gave the logarithms of all numbers, how could you find the logarithm of the product of several numbers? the logarithm of the quotient of one number by another number?

9. If you had a table which gave the logarithms of all numbers, and also the numbers corresponding to all logarithms, how could you find the product of two given numbers? the quotient of the first number by the second?

42. Logarithm of a Power. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent.*

If $x = \log A$, then $A = 10^x$.

Raising to the pth power, $A^p = 10^{px}$.

Hence $\log A^p = px$.

That is, $\boldsymbol{\log A^p = p \log A}$.

This is easily seen to be true for any base whatever, either by letting $A = b^x$ and proceeding as above, or by taking special numbers. Thus, if we take the base 2, instead of the base 10 as above, we have,

since $2^5 = 32$,

$\log_2 32 = 5$.

And since $(2^5)^2 = 32^2 = 1024$,

we have $\log_2 1024 = 2 \times 5$

$= 2 \log_2 32$.

That is, $\log_2 32^2 = 2 \log_2 32$.

43. Logarithm of a Root. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

If $x = \log A$, then $A = 10^x$.

Taking the rth root, $A^{\frac{1}{r}} = 10^{\frac{x}{r}}$.

Hence $\log A^{\frac{1}{r}} = \dfrac{x}{r}$.

That is, $\boldsymbol{\log A^{\frac{1}{r}} = \dfrac{\log A}{r}}$.

As in § 42, this can easily be proved for any base.

From §§ 42 and 43 we see that the raising of a number to any power, integral or fractional, reduces itself to multiplying the logarithm by the exponent, integral or fractional, and then finding the number of which the result is the logarithm.

Exercise 14. Powers and Roots

1. If we know that $\log 28.4 = 1.4533$, how can we find the logarithm of the fourth power of 28.4? What is this logarithm? What is the logarithm of 28.4^5?

As on page 30, we assume that we know this logarithm. The use of the table is explained a little later.

2. In Ex. 1, if we know that 650,000 is approximately the number whose logarithm is 5.8132, what do we know as to the approximate value of 28.4^4?

3. In our commercial work we shall occasionally need to find such a value as that of 1.05^7. If we know that $\log 1.05 = 0.0212$, and that 0.1484 is approximately the logarithm of 1.407, what is the approximate value of 1.05^7? State the reason.

4. In engineering work we occasionally need to find such a root as $\sqrt[4]{1.78}$. If $\log 1.78 = 0.2504$, what is $\log \sqrt[4]{1.78}$?

5. In Ex. 4, if we know that 0.0626 is the logarithm of 1.155, approximately, what do we know as to the value of $\sqrt[4]{1.78}$? State the reason.

6. If $x = 345 \times 1.005^{10}$, write the equation for $\log x$.

7. If $x = 65 \times \sqrt[5]{612}$, write the equation for $\log x$.

8. If $x = \dfrac{1}{1.06^4}$, write the equation for $\log x$.

9. Given that $\log 880 = 2.9445$ and $\log 1.05 = 0.0212$, find the value of $\log x$ from the equation $x = 880 \times 1.05^6$.

10. If we know that 1179 is approximately the number whose logarithm is 3.0717, what use in Ex. 9 can we make of this knowledge?

11. Write the equation for $\log A$ in the formula for the compound amount, $A = P(1+r)^n$.

44. Characteristic and Mantissa. Usually a logarithm consists of an integer plus a decimal fraction.

The integral part of a logarithm is called the *characteristic* and the decimal part is called the *mantissa.*

Thus if $\log 2353 = 3.37162$, the characteristic is 3 and the mantissa 0.37162. This means that $10^{3.37162} = 2353$, or that the 100,000th root of the 337,162d power of 10 is 2353, approximately.

45. Finding the Characteristic. Since we know that

$$10^3 = 1000 \quad \text{and} \quad 10^4 = 10{,}000,$$

we have

$$3 = \log 1000 \quad \text{and} \quad 4 = \log 10{,}000.$$

Hence the logarithm of a number between 1000 and 10,000 lies between 3 and 4, and is therefore 3 plus some fraction.

Similar reasoning shows us that the logarithm of a number between 10,000 and 100,000 lies between 4 and 5.

Likewise, since

$$10^{-2} = 0.01 \quad \text{and} \quad 10^{-3} = 0.001,$$

we have

$$-2 = \log 0.01 \quad \text{and} \quad -3 = \log 0.001.$$

Hence the logarithm of a number between 0.001 and 0.01 lies between -3 and -2, and is therefore equal to -3 plus some fraction. Hence the characteristic is -3.

Of course, instead of saying that log 1475 is 3 plus a fraction we might say that it is 4 minus a fraction; and instead of saying that log 0.007 is -3 plus a fraction we might say that it is -2 minus a fraction.

For convenience *the mantissa is always taken as positive,* although the characteristic may be either positive or negative.

The reason for this custom is apparent. We shall find (§ 49) that there are tables of mantissas of logarithms conveniently arranged for use. If we were compelled to use negative mantissas, the tables would be much more cumbersome and the work more complicated. Since we see that the characteristics are easily found without the aid of tables, the fact that they may be either positive or negative does not present any serious difficulties.

46. Rule for the Characteristic. From the reasoning set forth in § 45 we deduce the following rule:

1. *The characteristic of the logarithm of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, log 75 = 1 + some mantissa,
log 472.8 = 2 + some mantissa,
and log 14,800.75 = 4 + some mantissa.

2. *The characteristic of the logarithm of a number between 0 and 1 is negative, and is one more than the number of zeros between the decimal point and the first significant figure.*

For example, log 0.02 = − 2 + some mantissa,
and log 0.00076 = − 4 + some mantissa.

The logarithm of a negative number is an imaginary number, and hence such logarithms are not used in computation.

47. Negative Characteristic. If log 0.02 = − 2 + 0.30103, we cannot write the logarithm − 2.30103, because this would mean that both mantissa and characteristic are negative. It is often written $\overline{2}.30103$, which means that only the characteristic 2 is negative, the mantissa, 0.30103, being positive.

That is, $\overline{2}.30103 = -2 + 0.30103$. We may also write the logarithm 0.30103 − 2, or 8.30103 − 10, or in any other similar manner that will show clearly that the characteristic is − 2 and the mantissa 0.30103.

Exercise 15. Characteristics

Write the characteristics of the logarithms of:

1. 64.
2. 64.8.
3. 648.
4. 6.48.
5. 6480.
6. 6485.
7. 6485.7.
8. 6485.72.
9. 9235.
10. 923.57.
11. 92.35.
12. 9.235.
13. 92,350.
14. 0.9235.
15. 0.09235.
16. 70.
17. 0.7.
18. 0.07.
19. 0.77.
20. 0.773.

48. Mantissa independent of Decimal Point. It can be shown that

$$10^{3.37107} = 2350, \quad \text{whence} \quad \log 2350 = 3.37107.$$

Dividing by 10 we have

$$10^{3.37107-1} = 235, \quad \text{whence} \quad \log 235 = 2.37107.$$

Dividing each member of the first equation by 10^4, that is, by 10,000, we have

$$10^{3.37107-4} = 0.235, \quad \text{whence} \quad \log 0.235 = \bar{1}.37107.$$

That is, the mantissas are the same for log 2350, log 235, log 0.235, and so on, wherever the decimal point is placed.

The mantissa of the logarithm of a number is unchanged by any change in the position of the decimal point of the number.

This is a fact of great importance, for if the table of logarithms, which we shall soon describe, gives us the mantissa of log 235, we know that we may use the same mantissa for log 0.00235, log 2.35, and so on.

Exercise 16. Characteristics and Mantissas

Given log 734 = 2.8657, find the following:

1. log 73.4.
2. log 7.34.
3. log 0.734.
4. log 7340.
5. log 73,400.
6. log 0.0734.
7. log 734,000.
8. log 0.00734.
9. log 7,340,000.

Given log 42,360 = 4.6270, find the following:

10. log 4.236.
11. log 42.36.
12. log 423.6.
13. log 0.4236.
14. log 0.04236.
15. log 0.004236.
16. log 423,600.
17. log 4,236,000.
18. log 42,360,000.

Given log 2.705 = 0.4322, find the following:

19. log 27.05.
20. log 270.5.
21. log 0.2705.
22. log 0.02705.
23. log 2705.
24. log 270,500.

49. Table of Logarithms. The table of logarithms to six decimals on pages 231–248 shows the mantissas of all integral numbers from 100 to 9999, the decimal points being omitted.

In the table the first three figures of each number are given under N and the fourth figure under 0, 1, 2, $\cdots$, 9.

In the columns under 0 the mantissas are given to the full six decimals, the first two figures when omitted being the same as the first two figures of the mantissa above.

In the columns under 1, 2, 3, $\cdots$, 9 only the last four figures are given, and to these should be prefixed the first two figures of the mantissa in the same row and under 0.

Asterisks are used in place of 0's in certain cases to show that the first two figures of such mantissas should be increased by 1. This also applies to all mantissas in the same row and to the right of any mantissa which contains an asterisk.

Always write the characteristic before looking up the mantissa, so that the characteristic shall not be forgotten.

50. Finding the Logarithm of a Number from the Table.

1. Find the logarithm of 7.

Since the mantissa of log 7 is the same as that of log 7000, we look for 700 in column N and under column 0. Hence $\log 7 = 0.845098$.

2. Find the logarithm of 42,364.

First write the characteristic, 4.

Then notice that 42,364 is between 42,360 and 42,370, and is 0.4 of the way from 42,360 to 42,370. Hence we may assume that log 42,364 is approximately 0.4 of the way from log 42,360 to log 42,370.

In column N look for the first three figures, 423.

Then look to the right of 423 and in columns 6 and 7. Here we find that

$$\begin{aligned}\log 42370 &= 4.627058\\ \log 42360 &= \underline{4.626956}\\ 0.4 \text{ of } 0.000102 &= 0.000041\end{aligned}$$

Adding 0.000041 to 4.626956, we have $\log 42{,}364 = 4.626997$.

This plan of finding a logarithm is called *interpolation.*

3. Find the logarithm of 0.00007.

As in Ex. 1, $\log 7 = 0.845098.$

Hence $\log 0.00007 = \bar{5}.845098.$

4. Find the logarithm of 0.00027053.

First write the characteristic, − 4.
Proceeding as in Ex. 2, we have

$$\begin{aligned}\log 0.0002706 &= 0.432328 - 4\\ \log 0.0002705 &= \underline{0.432167 - 4}\\ 0.3 \text{ of } &\overline{0.000161} = 0.000048\end{aligned}$$

Adding 0.000048 to 0.432167 − 4, we have 0.432215 − 4.

We may write log 0.00027053 with the − 4 at the left, thus: $\bar{4}.432215$. When we have subtractions to perform, however, it is less confusing to place the negative characteristic at the right, as shown above. It is also convenient to write the negative characteristic at the right in performing other operations on logarithms.

Exercise 17. Finding Logarithms

Using the table on pages 231–248, find the logarithms of:

1. 35.
2. 43.
3. 863.
4. 575.
5. 907.
6. 2006.
7. 3000.
8. 3609.
9. 3656.
10. 36,567.
11. 36.567.
12. 365.67.
13. 700.
14. 7462.
15. 8174.
16. 5000.
17. 8000.
18. 0.9.
19. 0.06.
20. 0.743.
21. 0.799.
22. 2.444.
23. 2.775.
24. 27.75.
25. 3311.
26. 3312.
27. 3319.
28. 27,758.
29. 55,300.
30. 55,356.
31. 34,723.
32. 342.2.
33. 342.24.
34. 34,224.
35. 0.26.
36. 0.0026.
37. 1.05.
38. $1.05\frac{1}{2}$.
39. $1.04\frac{1}{2}$.
40. 0.3264.
41. 0.3798.
42. 3.3706.
43. 0.6600.
44. 0.6660.
45. 0.6666.
46. 0.66666.
47. 66,660.
48. 66,666.

51. Antilogarithm. The number corresponding to a given logarithm is called the *antilogarithm* of the logarithm.

Thus if log 676 is 2.829947, the antilogarithm of 2.829947 is 676.

52. Finding Antilogarithms. Antilogarithms may be found from the table of logarithms by looking for the number corresponding to the given mantissa, and locating the decimal point according to the characteristic.

1. Find the antilogarithm of 2.426836.

Looking for the mantissa 426836, we find that it is opposite 267 in column N and is in column 2. It is therefore the mantissa of 2672.

Since the characteristic is 2, there must be three integral places in the antilogarithm. Hence the antilogarithm is 267.2.

Therefore the antilogarithm of 2.426836 is 267.2.

2. Find the antilogarithm of $\bar{2}.840492$.

Looking for the mantissa 840492, we find that it lies between 840482 and 840545, their difference being 63. The difference between 840492 and 840482 is 10. Hence 840492 is $\frac{10}{63}$ of the way from 840482 to 840545.

Hence the antilogarithm of 840492 must be approximately $\frac{10}{63}$ of the way from the antilogarithm of 840482 to that of 840545.

We therefore write

$$\begin{aligned} \text{antilog}\ \bar{2}.840545 &= 0.069270\\ \text{antilog}\ \bar{2}.840482 &= \underline{0.069260}\\ \tfrac{10}{63} \text{ of } 0.000010 &= 0.0000016. \end{aligned}$$

Adding 0.0000016 to 0.069260, we have

$$\text{antilog}\ \bar{2}.840492 = 0.0692616.$$

For ordinary computations a six-place table is usually sufficient to give results that are accurate to six significant places.

3. Find the antilogarithm of 0.366487.

Looking for the mantissa 366487, we find that it lies between 366423 and 366610, their difference being 187. Since $366487 - 366423 = 64$, the given mantissa is $\frac{64}{187}$ of the way from 366423 to 366610. But antilog 0.366423 is 2.325, and antilog 0.366610 is 2.326. Adding $\frac{64}{187}$ of the difference between these logarithms to 2.325, we have 2.32534.

Exercise 18. Finding Antilogarithms

Using the table, find the antilogarithms of:

1. 0.478566.
2. 1.582177.
3. 3.670246.
4. $\bar{1}.737431$.
5. 0.577951.
6. 2.544316.
7. 4.589838.
8. 3.809694.
9. $\bar{2}.837083$.
10. $\bar{3}.874598$.
11. 1.999642.
12. 2.960557.
13. $\bar{2}.978646$.
14. 3.694653.
15. 0.000000.
16. 3.000000.
17. 4.740661.
18. 5.730628.
19. $\bar{2}.555168$.
20. $\bar{4}.652235$.
21. $\bar{2}.841087$.
22. 2.778556.
23. 2.362002.
24. 2.461098.
25. $\bar{1}.328046$.
26. $\bar{2}.866652$.
27. $0.278318 - 1$.
28. $0.686806 - 2$.
29. 4.981963.
30. 1.426582.

31. If the logarithm of the product of two numbers is 2.422918, what is the product of the numbers?

32. If the logarithm of the quotient of two numbers is 1.437592, what is the quotient of the numbers?

33. If the logarithm of the square of a number is 2.358616, what is the square of the number? What is the number?

34. If the logarithm of the square root of a number is 1.277942, what is the square root of the number? What is the number?

35. If we know that the logarithm of a result that we are seeking is 2.229581, what is the result?

36. There is a number such that the logarithm of its square is 2.306532. What is the number?

37. If the logarithm of the cube root of a number is 0.355219, what is the logarithm of the number? What is the cube root of the number?

53. Computation by Logarithms. Since we have learned (§§ 40–43) how to find the logarithms of products, quotients, powers, and roots, how to find (§ 50) the logarithms of all numbers, and (§ 52) how to find antilogarithms, we may now consider a few typical problems in computation.

1. Find the product of 247.6 and 3.954.

$$\begin{aligned}\log 247.6 &= 2.393751\\ \log 3.954 &= \underline{0.597037}\\ &\ 2.990788 = \log 979.011.\end{aligned}$$

If we multiply 247.6 by 3.954 the product is found to be 979.0104, or 979.010, if we carry the result to only six significant figures. Our last figure is therefore 1 too great. This tends to show us that computations by logarithms give, in general, results that are only approximately correct. The approximation is closer when we use larger tables. In all work in this book results should be carried to six significant figures only, and even with this limitation to six figures it should be understood that the last figure may not be correct.

2. Find the product of 37.24, 0.4163, and – 3.275.

Since we cannot find the logarithm of a negative number, we find the product of 37.24, 0.4163, and 3.275, and prefix the negative sign. We can do this because the product of two positive numbers is positive, and when this product is multiplied by a negative number, the result is negative.

$$\begin{aligned}\log 37.24 &= 1.571010\\ \log 0.4163 &= 0.619406 - 1\\ \log 3.275 &= \underline{0.515211}\\ &\ 2.705627 - 1\\ &= 1.705627 = \log 50.7723.\end{aligned}$$

Hence the product of 37.24, 0.4163, and – 3.275 is – 50.7723.

If we multiply in the ordinary way, the result is – 50.7723643 or, to six significant figures, – 50.7724. Hence the result by logarithms is correct to five significant figures and the sixth figure is 1 too great.

The great advantage arising from the use of logarithms is evident when we perform the above multiplication by the ordinary method. The advantage is even more apparent in Ex. 6.

3. Find the quotient of $17.28 \div 1.44$.

$$\begin{aligned} \log 17.28 &= 1.237544 \\ \log 1.44 &= \underline{0.158362} \\ &\ 1.079182 = \log 12, \text{ nearly.} \end{aligned}$$

Hence $17.28 \div 1.44 = 12$.

In this case we see that $\log 12 = 1.079181$, while the logarithm that is found is 1.079182, an error of 1 appearing in the sixth decimal place.

4. Find the quotient of $1.457 \div 544.5$.

$$\begin{aligned} \log 1.457 &= 0.163460 \\ \log 544.5 &= \underline{2.735998} \end{aligned}$$

Since we cannot subtract the larger logarithm from the smaller, we add an integral number to the characteristic of the logarithm of the dividend and also subtract the same number. In this case we may add and subtract 3. Then we have

$$\begin{aligned} \log 1.457 &= 3.163460 - 3 \\ \log 544.5 &= \underline{2.735998 \phantom{{}-3}} \\ &\ 0.427462 - 3 = \log 0.00267585. \end{aligned}$$

Hence $1.457 \div 544.5 = 0.00267585$.

The result by actual division is 0.00267585.

5. Find the value of 0.0048^3.

$$\begin{aligned} \log 0.0048 &= 0.681241 - 3 \\ &\ \underline{3\phantom{{}-3}} \\ &\ 2.043723 - 9 \\ &= 0.043723 - 7 = \log 0.000000110592. \end{aligned}$$

The result by actual multiplication is 0.000000110592.

6. Find the value of $\sqrt[3]{2.457 \times 3.829 \times 0.033176}$.

$$\begin{aligned} \log 2.457 &= 0.390405 \\ \log 3.829 &= 0.583085 \\ \log 0.033176 &= \underline{0.520824 - 2} \\ &\ 1.494314 - 2 = 2.494314 - 3 \end{aligned}$$

We avoid fractions in the characteristic by writing $2.494314 - 3$ instead of $1.494314 - 2$.

Then $\frac{1}{3}$ of $(2.494314 - 3) = 0.831438 - 1 = \log 0.678325$.

Hence $\sqrt[3]{2.457 \times 3.829 \times 0.033176} = 0.678325$.

Exercise 19. Computations by Logarithms

Perform the following computations by logarithms:

1. 2.274×38.58.
2. 6.574×3245.
3. 20.72×80.36.
4. 4062×3186.
5. 372×5765.
6. 697×389.88.
7. 26.2×38.813.
8. 63.5×4.2764.
9. 5.6×3.2725.
10. 6.3×0.0246.
11. 1.02^5.
12. 1.03^7.
13. 1.04^9.
14. 1.05^{15}.
15. $(1.04\frac{1}{2})^4$.
16. $(1.03\frac{1}{2})^8$.
17. $8.006 \div 3.9$.
18. $9 \div 4.1425$.
19. $2 \div 21.458$.
20. $0.8 \div 2.152$.
21. $\sqrt{2}$.
22. $\sqrt[3]{2}$.
23. $\sqrt[4]{7}$.
24. $\sqrt[5]{8}$.
25. $\sqrt[6]{128}$.
26. $\sqrt[7]{147.6}$.
27. $\sqrt[8]{0.0005}$.
28. 22.352^7.
29. 6.2077^{13}.
30. 0.0084^3.

31. Find the value of $\sqrt{3.245 \times 22.956 \times 814.87}$.

32. Find the value of $\sqrt[3]{0.6 \times 0.03654 \times 298.48}$.

33. Find the value of $6.76 \times 39.352 \times 42.967 \times 0.08$.

Perform the following multiplications:

34. 6.289×0.0006.
35. 36.756×0.00009.
36. 0.875×0.0072343.
37. -68.26×27.654.
38. $-37.854 \times (-3.695)$.
39. $-4.6789 \times (-0.00832)$.

Perform the following divisions:

40. $\dfrac{37.757}{82.858}$.
41. $\dfrac{0.49888}{2.4265}$.
42. $\dfrac{676.39}{0.004345}$.
43. $\dfrac{0.091945}{37.997}$.
44. $\dfrac{0.83956}{0.79224}$.
45. $\dfrac{0.00088743}{0.62889}$.

Perform the following operations:

46. $35^{\frac{1}{3}}$. **47.** $5.2^{\frac{2}{3}}$. **48.** $144^{\frac{3}{2}}$. **49.** $15^{\frac{3}{4}}$.

54. Cologarithm. The logarithm of the reciprocal of a number is called the *cologarithm* of the number.

The cologarithm of x is expressed thus: $\operatorname{colog} x$.

Since $\operatorname{colog} x = \log \frac{1}{x} = \log 1 - \log x = 0 - \log x$, it is evident that

$$\mathbf{colog}\ x = -\ \mathbf{log}\ x.$$

We may write this $10 - \log x - 10$, thus avoiding negative mantissas. Hence

$$\operatorname{colog} 2 = 10 - \log 2 - 10 = 10 - 0.301030 - 10 = 9.698970 - 10.$$

We may write this $0.698970 - 1$, $\bar{1}.698970$, or $9.698970 - 10$.

55. Use of the Cologarithm. Since to divide by a number is the same as to multiply by its reciprocal, *instead of subtracting the logarithm of a number we may add its cologarithm.*

The cologarithm is easily written by looking at the logarithm in the table. Thus, since $\log 21 = 1.322219$, to find $\operatorname{colog} 21$ we subtract this from $10.000000 - 10$, or from $9.99999\,(10) - 10$. That is, beginning at the left we subtract from 9 the number represented by each figure, except the right-hand one, which we subtract from 10. In full form we have

$$\begin{array}{rll}
 & 10.000000 - 10 & = 9.99999\ (10) - 10 \\
\log 21 = & \underline{1.322219} & = \underline{1.32221\ \ \ 9} \\
\operatorname{colog} 21 = & & \ 8.67778\ \ \ 1\ - 10 = \bar{2}.677781
\end{array}$$

Similarly, we may find $\operatorname{colog} 0.03952$ thus:

$$\begin{array}{rl}
 & 9.99999\ (10) - 10 \\
\log 0.03952 = & \underline{8.59681\ \ \ 7\ \ - 10} \\
\operatorname{colog} 0.03952 = & 1.40318\ \ \ 3 \qquad = 1.403183
\end{array}$$

Practically, of course, we would look at $\log 0.03952$ and subtract mentally, writing 1.403183 at once.

For example, we see that the cologarithms given below are nearly as easily written as the logarithms:

Logarithms,	3.604257,	5.960506,	7.431682 − 10.
Cologarithms,	6.395743 − 10,	4.039494 − 10,	2.568318.

If we have to subtract a single logarithm, it is simpler not to use the cologarithm; but in complicated operations, as on page 44, it is easier to use cologarithms.

56. Advantages of the Cologarithm. The advantages of the cologarithm may be seen in simplifying the following expression:

$$\frac{17.28 \times 6.25 \times 16.9}{1.44 \times 0.25 \times 1.3}.$$

This case has been selected because it is one in which we can easily verify the answer by cancellation. By logarithms we have

$$\begin{aligned}
\log 17.28 &= 1.237544\\
\log 6.25 &= 0.795880\\
\log 16.9 &= 1.227887\\
\operatorname{colog} 1.44 &= 9.841638 - 10\\
\operatorname{colog} 0.25 &= 0.602060\\
\operatorname{colog} 1.3 &= \underline{9.886057 - 10}\\
&\ 23.591066 - 20\\
&= 3.591066 = \log 3900.
\end{aligned}$$

Exercise 20. Computations by Logarithms

Perform the following computations by logarithms:

1. $\dfrac{48{,}000}{1.05^4}$.

2. $\dfrac{27.8 \times 0.44}{3.25 \times 4.2}$.

3. $\dfrac{62.8 \times 8.345}{34.9 \times 0.65}$.

4. $\left(\dfrac{0.448 \times 6.98}{49.6 \times 9.76}\right)^{\frac{1}{2}}$.

5. $\left(\dfrac{11.73 \times 4.025}{7.46 \times 69.2}\right)^{\frac{1}{4}}$.

6. $\sqrt{\dfrac{7 \times 2.1 \times 8.2}{0.6 \times 5.9 \times 4.3}}$.

7. $\sqrt{\dfrac{33 \times 9.8 \times 6.8}{4.9 \times 2.7 \times 0.8}}$.

8. $\sqrt[3]{\dfrac{61 \times 3.8 \times 994.26}{44.8 \times 8.1235 \times 4.7}}$.

9. $\sqrt[3]{\dfrac{26 \times 47 \times 928.84}{33.5 \times 71.068 \times 8.7}}$.

10. $\left(\dfrac{37.645 \times 49.42 \times 81.8}{62.7 \times 0.8 \times 36.496}\right)^{\frac{2}{3}}$.

11. $\sqrt[4]{\dfrac{4.2 \times 2.8 \times 3.46}{1.42 \times 5.46 \times 86.078}}$.

12. $\sqrt[7]{\dfrac{8 \times 9.1 \times 0.22354 \times 66}{1.7 \times 8.82 \times 0.91}}$.

13. $\sqrt[8]{\dfrac{8 \times 72 \times 676 \times 0.53152}{4.7 \times 31.8 \times 275}}$.

14. $\sqrt[10]{\dfrac{2436 \times 37632 \times 42956}{74188 \times 99065 \times 829.7}}$.

Exercise 21. Review of Chapter II

Using the base 10, express as logarithmic equations:

1. $10^3 = 1000.$ **3.** $10^{2.4771} = 300.$ **5.** $10^{-2} = 0.01.$

2. $10^{3.5843} = 3840.$ **4.** $10^{2.5051} = 320.$ **6.** $10^{-3} = 0.001.$

Write the equation expressing the value of log x from each of the following equations:

7. $x = 329 \times 1.06^5.$ **8.** $x = 1.045^{11}.$ **9.** $x = 1 \div 1.035^7.$

Express as exponential equations:

10. $\log 5211 = 3.7169.$ **12.** $\log 9.072 = 0.9577.$

11. $\log 48 = 1.6812.$ **13.** $\log 0.0001 = -4.$

Find the value of x in each of the following equations:

14. $\log x = 2.434249.$ **15.** $\log x = 4.897902 - 10.$

16. Given that $x = 345 \times 1.045^6 \div 7$, find the value of x.

17. Given that $x = \sqrt[5]{7963}$, find the value of x.

18. Given $r = 34.65$ in. and $\log \pi = 0.497151$, find the area of the circle, πr^2.

19. A bin 11 ft. 8 in. long, 6 ft. wide, and 5 ft. 6 in. deep is filled with wheat. Given that 1 bu. = 2150.42 cu. in., find by logarithms the value of the wheat at $2.12 a bushel.

20. Given that $5^{x+1} = 40$, find the value of x.

21. Find the value of the expression

$$\frac{35 \times 8.2^5 \times 3^4 \times \sqrt[4]{71}}{7 \times 3^{11} \times \sqrt[3]{94}}.$$

22. Using logarithms to find the value of 1.02^{-5}, find the value of the expression $5\left(1 - \frac{1}{1.02^5}\right)$.

Exercise 22. Review of Chapters I and II

1. Given that $1.04^7 = 1.3159$ and $1.04^4 = 1.1699$, find the value of 1.04^{11} and state the law of exponents used.

By the aid of the Binomial Theorem, find the values of:

2. 1.02^2.
3. 1.03^2.
4. 1.04^2.
5. 1.05^2.
6. 1.06^2.
7. 1.07^2.
8. 1.01^3.
9. 1.03^3.
10. 1.04^3.
11. 800×1.08^2.
12. 500×1.08^2.
13. 950×1.05^3.

14. Find the value of 9400×1.045^{11}, using either logarithms or the Binomial Theorem, but choosing the simpler of the two methods.

15. By the method of § 25 and also by the aid of logarithms, find to one decimal place the value of $\sqrt{8456}$.

16. Find the value of $0^{10} + 16^{\frac{1}{2}} + 21^0 + 1^{-1} - 4^{\frac{3}{2}}$.

17. Given that $x = 21^{-7}$, find the value of $\log x$; then find the value of x.

18. Given that $x = \sqrt{71} \div \sqrt[3]{45}$, find the value of x.

19. Change the fraction $3\,a^5b^{-2}c/2^{-1}x^3y^2z^{-5}$ to an expression in which all the exponents are positive.

20. Multiply $2\,x^{-2} + 5\,x^{-1}y + 3\,y^2$ by $5\,x^{-1} + 2\,y$, and check the work by division.

21. Find to two decimal places the value of $\sqrt[7]{231}$.

22. Given that $10^x = 341$, find the value of x.

23. Given that $\log x = 2.313986$, find the value of $\sqrt{x}$.

24. Given that $\log x = 1.357902$ and $\log y = 2.557548$, find the value of xy.

25. Given the formula found in physics for the law of falling bodies, $S = \frac{1}{2}\,gt^2$, in which $g = 32.16$ ft., find by logarithms the value of S when $t = 21$.

CHAPTER III

THE SLIDE RULE

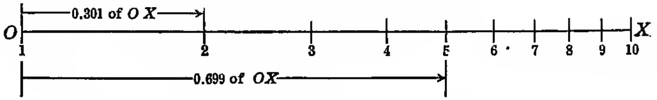
57. Nature of the Slide Rule. Since we can easily multiply one number by another by the simple expedient of adding logarithms, and since we can add two lines by properly joining one to the other, we are naturally led to consider the possibility of multiplying by means of adding lengths. Practically, this is done by means of two rulers, one of which slides along the other. The rulers are marked to show logarithms of numbers, and by adding these logarithms we readily find the logarithm of the product, and hence the product itself. As in the case of logarithms, the slide rule can also be used for division, and for finding powers and roots.

58. Accuracy of the Work. Since all measurements are only approximately accurate, we practically need to have the results of computation relating to measurements only approximately accurate. That is, if we make a certain measurement accurate within 0.01 in., no computation based upon it need be carried beyond hundredths of an inch, but the computation must be accurate to that point. Logarithms and the slide rule give only approximate results, but results that are accurate within certain required limits.

The slide rule is used very extensively by mechanics and engineers, but it also has a definite use in commercial work, affording one of the most convenient checks on various operations. Even if the slide rule used is so small as to give only three figures accurately, this may suffice for checking.

59. Numbers on a Slide Rule. In a table of logarithms we need the mantissas only to find the mantissa of the result, and we do not need that mantissa except to find the result. Hence on a slide rule we need not print the mantissas, provided we print the numbers to which they correspond. The numbers represent the numbers with which we are working, the mantissas of their logarithms, which we shall speak of briefly as their mantissas, being represented simply by marks.

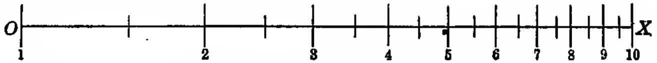
For example, we may represent the mantissas of numbers from 1 to 10 by marks (or lengths), thus:



Here $O2$ is 0.301 of OX, because log 2 is 0.301, to three decimals. Similarly, $O5$ is 0.699 of OX, because log 5 is 0.699. That is, the lengths $O2$, $O3$, $O4$, $\cdots$, are the logarithms of 2, 3, 4, $\cdots$. Furthermore, since the mantissas are the same for the logarithms of 20, 30, 40, $\cdots$ as for 2, 3, 4, $\cdots$, we can use the lengths $O2$, $O3$, $O4$, $\cdots$ for the mantissas of 20, 30, 40, $\cdots$, 200, 300, 400, $\cdots$, and so on.

The decrease in the lengths of the lines from left to right corresponds exactly to the decrease in the differences between the logarithms from 1 to 10. Thus, $\log 1 = 0$, $\log 2 = 0.301$; $\log 3 = 0.477$, $\log 4 = 0.602$, $\cdots$, the differences being 0.301, 0.176, 0.125, $\cdots$.

We may also insert marks to show the mantissas for the logarithms of 1.5, 2.5, 3.5, $\cdots$, thus:



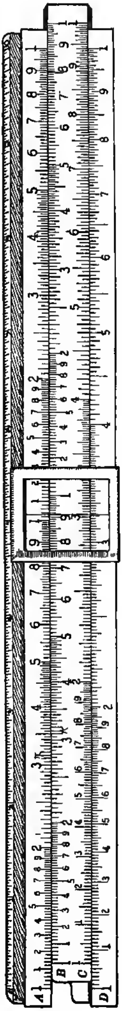
Since log 1.5 is 0.176, which is not exactly half the difference between log 1 and log 2, the line of division does not bisect $O2$, and similarly for the other cases.

60. Slide Rule. The form here shown is a common form of the slide rule.

The rule in the middle slides along the two outer rules. The glass plate in the middle, known as the *runner* or *cursor*, slides either way, a hair line being ruled upon it to facilitate reading the results.

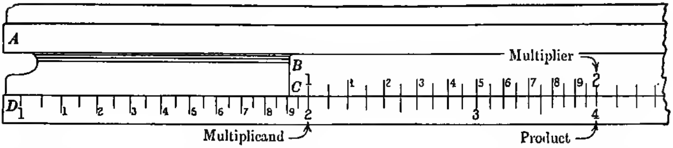
Scales *A* and *B* are alike, the length from 1 to 2 being log 2, that from 1 to 3 being log 3, and so on. Each scale is duplicated, the left half running from 1 to 10, and the right half also running from 1 to 10, for reasons explained later. Scales *A* and *B* are used when, owing to the different graduation of *C* and *D*, the latter are not convenient.

Scales *C* and *D* are also alike, but they are not the same as *A* and *B*. If you measure the distance from 1 to 2 (not the tenths) on *C*, you will see that it is twice the distance from 1 to 2 on the *A* or *B* scale; indeed, the 2 on *C* is exactly below the 4 on *B*. Hence on *C* and *D* it is easy to find the length corresponding to the mantissa of log 125, to judge the mantissa of log 1255 very closely, and even to judge the mantissa of the logarithm of such a number as 1257 with a fair degree of accuracy.



It is needless to say that no explanation of a slide rule given in a book is at all complete. No one can learn to use a slide rule without having one to work with, and no written explanation is ever as satisfactory as one given by an instructor with an instrument at hand. Large slide rules that can be read across the room can be obtained for purposes of instruction. The student should construct a rule, using paper and marking the main divisions.

61. Multiplication. Suppose that we wish to multiply one number by another, taking for purposes of illustration the case of 2 × 2. We place 1 on *C* exactly over 2 on *D*. We then look for 2 on *C* and find that it is exactly over 4, and hence we know that 4 is the product of 2 and 2.



Similarly, on the same figure, the product of 2×1.5 is found just below the 5 which is between 1 and 2; that is, just below 1.5. Hence we see that $2 \times 1.5 = 3$. We have here added mechanically the mantissa of 2 and that of 1.5, the resulting mantissa corresponding to the number 3. In the same way, from the same figure, we can read such products as the following:

$2 \times 1.1 = 2.2$ $\quad 2 \times 1.3 = 2.6$ $\quad 2 \times 1.6 = 3.2$

$2 \times 1.2 = 2.4$ $\quad 2 \times 1.4 = 2.8$ $\quad 2 \times 1.9 = 3.8$

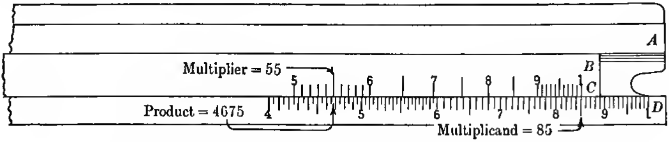
The whole scheme of multiplication is easily seen by a study of the following cases:

C	1	2	*C*	1	1.2	*C*	1	2.8
D	2	4	*D*	1.5	1.8	*D*	2.1	5.88
$2 \times 2 = 4$			$1.2 \times 1.5 = 1.8$			$2.8 \times 2.1 = 5.88$		

That is, *place the 1 on C over the multiplicand on D, and read the product on D below the multiplier on C.*

In setting the left-hand index or unit of *C* over the multiplicand, if we find the multiplier off the rule we merely move the slide to the left, setting the right-hand index over the multiplicand. We then read the product under the multiplier. This has the same effect as repeating the *D* scale under the *C* scale which protrudes to the right.

62. Number of Integral Places in the Product. We have thus far left the student to determine by his common sense the number of integral places in a product. It is, however, desirable to have a definite method of determining this number. The number of integral places manifestly depends on the characteristic of the logarithm of the result, and this is equal to the sum of the characteristics of the logarithms of the numbers, plus anything that is carried over in adding the mantissas. To show how the sum of the mantissas may exceed 1, consider the product of 55 × 85.



If we set the unit on C over 55 on D, or over 85 on D, the 85 on C or 55 on C will be off the D-scale to the right. This means that the two mantissas together are equal to more than a unit, and the characteristic of the logarithm of the product will be $(1+1)+1$ carried from the mantissas. There will thus be four integral places in the result.

Hence, *each time that it is necessary to move the slide to the left instead of to the right, add 1 to the characteristic.*

Thus, in the case of 250 × 285 the sum of the characteristics is 4. Setting the slide, we can read the result with the slide projecting to the right, and so the characteristic is 4 and there are five integral places in the result. We read the result on D under 285 on C to be 71,250.

But in the case of 35 × 850, in which the characteristics are 1 and 2, by setting the left unit on C over 35 on D we find that the sum of the mantissas of 35 and 850 is more than the length of the rule, or more than 1. The characteristic of the result is therefore $1 + 2 + 1 = 4$, and the result will have five integral places. Placing the right-hand unit on C over 850 on D, going to 35 on C, and dropping down to D we find the result to be 29,750.

Exercise 23. Multiplication

1. A cubic foot of water weighs 62.5 lb. and the specific gravity of a certain grade of steel is 8. Find the weight of 1 cu. ft. of this grade of steel.

When we say that the specific gravity of steel is 8, we mean that steel is 8 times as heavy as water. The weight of 1 cu. ft. of water should be memorized.

The problem requires the multiplication of 62.5 by 8. Place the right-hand 1 on *C* over 6.25 on *D* and read the result 500 on *D* just below the 8 on *C*, making due allowance for the decimal point.

2. The specific gravity of a certain grade of cast iron is 7.2. Find the weight of 1 cu. ft. of this grade of cast iron.

Using the slide rule, perform these multiplications:

3. 4 × 8.
4. 4 × 80.
5. 4 × 87.
6. 4 × 8.7.
7. 0.4 × 0.87.
8. 2.1 × 3.7.
9. 3.4 × 2.8.
10. 2.9 × 6.2.
11. 1.8 × 7.8.
12. 3.4 × 6.9.
13. 1.25 × 36.
14. 1.25 × 4.8.
15. 1.15 × 2.9.
16. 2.35 × 4.65.
17. 4.25 × 8.75.

18. Find the interest on $1500 for 1 yr. at 6%.

19. Find the interest on $2500 for 1 yr. at 4%.

20. Find the discount on $750 at 15%.

Using the slide rule, find the discounts on the following:

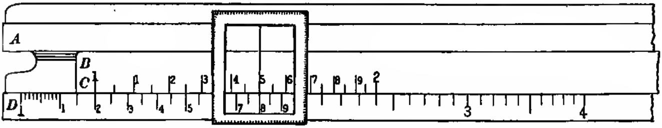
21. $535 at 8%.
22. $725 at 6%.
23. $645 at 5%.
24. $885 at 4%.
25. $1250 at 12%.
26. $2700 at 16%.

27. Find the premium on a life-insurance policy of $5000 at $21.70 per thousand.

Multiply 2.17 by 5, moving the decimal point to meet the conditions.

28. Find the premium on a fire-insurance policy of $7500 at $1.24 per hundred.

63. Continued Multiplication. By using the runner we are able to perform continued multiplication or division without reading the intermediate products or quotients. For example, suppose that we wish to find the product of 12 × 15 × 20.



In the illustration the minor subdivisions are omitted. We set the unit on C above 12 on D and place the runner so that the hair line crosses 15 on C. Now, instead of reading the result, bring 1 on C exactly under the hair line, go to 2 on C, drop down, and read the result 3600. Since the characteristics of 12, 15, and 20 are 1, 1, 1, and since the slide was moved continuously to the right, there is nothing to carry. Hence the characteristic is 3, and there are four integral places in the product.

The student must use his judgment as to how many significant figures to seek in the result, depending on the size of the slide rule used.

Exercise 24. Continued Multiplication

1. In the interest formula $i = prt$ find the value of i when $p = 750$, $r = 0.035$, and $t = 2.5$.

2. A bill of goods amounting to $1250 is allowed discounts of 10 and 12. Find the net amount.

Notice that we need to find 0.9 × 0.88 × $1250.

3. Find the cost of 18 shipments of flour, 75 bbl. each, at $9.50 per barrel.

Using the slide rule, perform these multiplications:

4. 3 × 4 × 5.

5. 32 × 40 × 50.

6. 3.4 × 4.6 × 50.

7. 3.8 × 4.5 × 57.

8. 47 × 56 × 93.

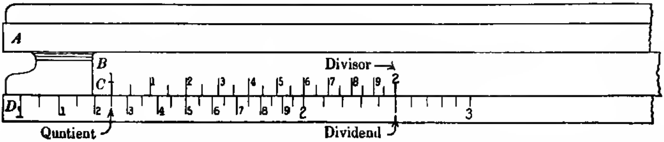
9. 2 × 17 × 175.

10. 3 × 26 × 38.4.

11. 0.7 × 4.8 × 142.

64. Division. Since the logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor, we perform division on the slide rule by simply reversing the multiplication. We pick out the dividend on *D*, set the divisor above it on *C*, and read the quotient under the unit of *C*, to the right or to the left as the case may be.

1. Required to divide 25 by 2.



Placing the divisor 2 on *C* over the dividend 25 on *D* we read the quotient 125 on *D* under the unit on *C*. Since it is evident where the decimal point should be placed, we write 12.5 as the result.

2. Required to divide 350 by 55.

Placing 55 on *C* over 350 on *D*, under the right-hand unit on *C* we read the quotient 6364 on *D*. Since the mantissa of 55 exceeds that of 350, as is shown by its position on the rule to the right of 350, we must decrease the characteristic of log 350 by 1. This leaves 0 as the characteristic of the result, so that the result is 6.36.

For brevity we may speak of the characteristic of a number, meaning the characteristic of the logarithm of the number.

65. Number of Integral Places in the Quotient. When the slide projects to the right in division, we subtract the characteristic of the divisor from that of the dividend to find the characteristic of the quotient. When the slide projects to the left, we reduce this characteristic by 1.

Required to divide 47.5 by 0.675.

Placing 675 on *C* above 475 on *D*, under the right-hand unit on *C* we read the quotient 704. The difference of the characteristics is $1-(-1)$, or 2. Since the slide projected to the left we have $2-1=1$, the characteristic of the quotient. Hence the result is 70.4.

Exercise 25. Division

1. Find the diameter of a cast-iron water pipe whose circumference is 27 in.

Here we have to divide 27 by π, and it will be observed that the number π is given on many slide rules. In case the student has a slide rule on which it is not given, he may use $3\frac{1}{7}$ or 3.14 as the value.

2. In Ex. 1 suppose the circumference to be 28 in.

3. If 12 workmen together receive $33, what is the average amount received by each?

4. If the interest on $1200 for 1 yr. is $69, what is the rate of interest?

5. If the discount on a bill of goods for $1500 is $120, what is the rate of discount?

6. If the commission on a sale of $4500 worth of goods is $135, what is the rate of commission?

7. If 7500 yd. of a certain grade of cloth cost $6000, what is the cost of the cloth per yard?

Perform the following divisions:

8. 4 ÷ 2.
9. 2 ÷ 4.
10. 42 ÷ 21.
11. 21 ÷ 63.
12. 75 ÷ 2.5.
13. 12.5 ÷ 2.5.
14. 17.28 ÷ 12.
15. 1.728 ÷ 1.2.
16. 6250 ÷ 25.
17. 312.5 ÷ 2.5.
18. 1.584 ÷ 12.
19. 15.84 ÷ 1.2.
20. 3885 ÷ 35.
21. 38.85 ÷ 3.5.
22. 6970 ÷ 41.
23. 70.11 ÷ 4.1.
24. 9300 ÷ 6.2.
25. 5.329 ÷ 73.

Perform the following divisions, carrying each result to three significant figures:

26. 2 ÷ 3.
27. 3 ÷ 7.
28. 5 ÷ 6.
29. 23 ÷ 6.
30. 41 ÷ 23.
31. 37 ÷ 17.
32. 1.5 ÷ 0.7.
33. 2.8 ÷ 13.
34. 611 ÷ 1.9.

66. Continued Multiplication and Division. In commercial work we sometimes need to make such calculations of continued multiplication and division as the following:

$$\frac{4.25 \times 67.3 \times 300}{583 \times 0.07}.$$

In such cases the slide rule will be found very helpful.

Using the hair line of the runner, we first perform the continued multiplication of the numerator, as explained in § 63, placing the hair line over the product of the three numbers. This result is then divided by 583, and the result of this division is again divided by 0.07. The figures of the result are 2, 1, 0. In performing the multiplications, we notice that the slide projected to the left in finding the first product but not in finding the second product. In division, the slide projects to the left only when dividing by 0.07. Now every time the slide projects to the left in multiplying, the characteristic of the result is increased by 1; and every time it projects to the left in dividing, the characteristic of the result is decreased by 1. Hence in this case the characteristic is changed $1-1$, or 0, and we have:

Characteristic of log 4.25 is	0
Characteristic of log 67.3 is	1
Characteristic of log 300 is	2
Characteristic in the numerator is	3

The characteristic in the denominator is evidently $2+(-2)$, or 0. Hence the characteristic of the result is 3, and there are four integral places. The result is, therefore, 2100.

The results are valuable as checks and are often quite sufficient for practical purposes. In continued multiplication and division place at the head of two columns the signs × and ÷ respectively. Then place a check in the proper column every time the slide projects to the left. These checks will be found to be a convenient aid to the memory.

Exercise 26. Multiplication and Division

1. The pull P of a locomotive is given by the formula $P = d^2ps/D$, where d is the diameter of each cylinder in inches, p the pressure of steam in pounds per square inch, s the stroke in inches, and D the diameter of the drive wheels in inches. Find the value of P when $d = 10.5$, $p = 140$, $s = 18$, and $D = 36$.

2. For a train going at the rate of s feet per second round a curve of radius r feet the outer rail should be raised h inches above the inner, where $h = s^2G/386\,r$, in which G is the number of feet in the gauge of the track. Taking $s = 60$, $r = 2700$, $G = 4.7$, find the value of h.

Simplify the following:

3. $\dfrac{2 \times 3 \times 64}{4 \times 6 \times 8}$.

4. $\dfrac{3 \times 30 \times 49}{5 \times 7 \times 9}$.

5. $\dfrac{45 \times 72 \times 7}{40 \times 81}$.

6. $\dfrac{45 \times 49 \times 11}{35 \times 63}$.

7. $20 \times 75 \times 45 \times 375$.

8. $\dfrac{120 \times 55 \times 4}{200 \times 22}$.

9. $\dfrac{75 \times 82.5}{12.5 \times 37.5}$.

10. $\dfrac{15.6 \times 24 \times 3.9}{1.44 \times 16.9}$.

11. $\dfrac{2 \times 7.35 \times 8.25}{1.05 \times 7}$.

12. $31.2 \times 7.2 \times 12.4 \times 4.75$.

Find the first three significant figures in each of the following:

13. $\dfrac{28 \times 39 \times 7.3}{21 \times 61}$.

14. $\dfrac{35 \times 72 \times 8}{67 \times 4.9 \times 3}$.

15. $1.215 \times \$850 \div 7$.

16. $\dfrac{436 \times 500 \times 8}{42 \times 66 \times 320}$.

17. $\dfrac{3.4 \times 126 \times 98.7}{7.2 \times 8.4 \times 6.9}$.

18. $1.04 \times \$2500 \div 6$.

67. Proportion. One of the characteristic features of the slide rule is readily seen to be that, however the slide is placed, all numbers on the slide are proportional to the numbers on the rule that coincide with them. That is, when 2 is over 4 it will be seen that 3 is over 6, 4 is over 8, 2.5 is over 5, 1.2 is over 2.4, and so on; in other words,

$$3:6 = 4:8 = 2.5:5 = 1.2:2.4 = \cdots.$$

Expressed diagrammatically,

C	Set first term	Then under third term
D	Over second term	Read the fourth term

In the proportion $1.5:3.7 = 4.1:x$ find the value of x.

Setting 1.5 over 3.7, we read 10.1 under 4.1. In this case we use the *A* and *B* scales, since the result exceeds 10, the limit of the *D* scale.

Exercise 27. Proportion

1. Show that the rule for proportion includes the rules for multiplication and division as special cases.

2. Given that 1 kilogram is equal to 2.2 lb., set the slide so as to read the pounds corresponding to kilograms.

3. Given that 1 meter is equal to 39.37 in., set the slide so as to read the inches corresponding to meters.

4. A map is drawn to the scale 1 in. = 50 mi. Set the slide so as to read the miles corresponding to inches.

5. If the wages of 7 men for 1 da. is $24.50, find the wages of 11 men at the same rate.

6. If a factory can turn out 3240 pairs of shoes in 6 da., how long will it take to fill an order for 25,000 pairs?

In all such cases it is understood that the same rate is maintained.

7. Given that the rate of wages is $25 per week of 44 hr., set the slide rule to the proper ratio and check the following items of a payroll, making the necessary corrections:

NUMBER	HOURS	WAGES		CORRECTION	
637	41	23	30		
638	38	21	48		
642	44	25			
675	37	21	12		
676	42	23	85		
677	35	19	80		
679	43	24	61		
680	36	20	46		
681	39	22	15		

8. At a special sale a certain grade of ribbon was marked 12¢ a yard, or 3 yd. for 35¢. Set the *A* and *B* scales so that the cost of any number of yards can be read by using the cheaper of these two rates.

9. Allowing 2150 cu. in. to the bushel, set the *A* and *B* scales to read the number of bushels corresponding to any number of cubic inches. Then find the number of bushels in 52,500 cu. in.

10. If 15 workmen are assigned to the work on a given order for castings, they can complete the job in $3\frac{1}{4}$ da. Working at the same rate, how many men must be put on the job to cut the time down to 2 da. 7 hr.?

Take 2 da. 7 hr. as equal to 2.87 da., which is approximately the case with working days of 8 hr. each.

11. Using the scale of 14 gal. = 53 liters, set the slide rule to find the number of gallons in any given number of liters. Then find the number of gallons in 60 liters; in 90 liters.

68. Changing a Fraction to a Decimal. To change a common fraction to a decimal fraction is merely to divide the numerator by the denominator.

Thus, to change $\frac{13}{64}$ to a decimal we have to divide 13 by 64. Placing 64 of *C* over 13 of *D*, we read the result on *D* under 1 (or 10) on *C*. Hence we find that $\frac{13}{64} = 0.203$.

69. Finding a Rate Per Cent. To find what per cent one number is of another is merely a case of division.

Thus, 7 is $\frac{7}{12}$ of 12, and the slide rule shows that $\frac{7}{12} = 0.583$.

70. Interest. Since $i = prt$, it is evident that the calculation of interest can be checked on the slide rule, the operation being merely continued multiplication.

Thus, to find the interest on $850 for 83 da. at 5%, we have

$$i = \frac{850 \times 0.05 \times 83}{360}.$$

Using the slide rule and performing the four operations in order, we find that the interest is $9.80. In this case the interest was found exactly, and, in general, the approximation is sufficient for a check.

71. Inverse Cases in Interest. Since $i = prt$, it is evident that the slide rule is easily adapted to finding p from the relation $p = i/rt$, and similarly to finding r and t.

Thus, to find the principal which will gain $250 interest in 7 yr. at 5½%, we have

$$p = \frac{250}{0.055 \times 7}.$$

Using the slide rule and performing the operations in order, we find that the principal is approximately $650.

To use the slide rule in finding the value of p when a, r, and t are given, we derive the formula

$$p = \frac{a}{1 + rt}.$$

The divisor $1 + rt$ can usually be calculated mentally.

Exercise 28. Miscellaneous Applications

Change the following common fractions to decimals:

1. $\frac{7}{16}$.	**4.** $\frac{17}{24}$.	**7.** $\frac{17}{48}$.	**10.** $\frac{9}{64}$.	**13.** $\frac{7}{72}$.
2. $\frac{9}{16}$.	**5.** $\frac{19}{32}$.	**8.** $\frac{23}{48}$.	**11.** $\frac{19}{64}$.	**14.** $\frac{5}{72}$.
3. $\frac{5}{24}$.	**6.** $\frac{29}{32}$.	**9.** $\frac{41}{48}$.	**12.** $\frac{37}{64}$.	**15.** $\frac{17}{72}$.

16. The following table, compiled from a random selection of a few workmen in several states, shows the relative number of workers in the various wage classes in the lumber and furniture business, with their wages. Fill the vacant spaces in the table, performing all calculations of multiplication and division with the slide rule.

STATE	16¢		18¢		20¢		25¢		30¢		40¢		50¢		TOTAL NUMBER
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	
Georgia	29	12.5	31	13.4	4		68		74		26		—	—	232
Illinois	—	—	—	—	—	—	51		75		269		80		
Mass.	—	—	2		10		48		44		87		41		
New York	—	—	6		8		175		187		252		38		
Wisconsin	127		372		139		272		93		19		—	—	

The numbers 16¢, 18¢, . . . , mean that the wages were 16¢ or less per hour. Thus, 29 workmen out of 232, chosen at random in Georgia, received 16¢ or less per hour; that is, 12.5% received this wage.

Calculate the interest on the following:

17. \$850, 145 da., 4%.

18. \$700, 77 da., $4\frac{1}{2}$%.

19. \$900, 65 da., $3\frac{1}{2}$%.

20. \$850, 73 da., $5\frac{1}{4}$%.

21. \$975, 2 yr. 3 mo., 5%.

22. \$7850, 87 da., $4\frac{1}{4}$%.

23. \$6500, 77 da., $3\frac{1}{4}$%.

24. \$7500, 21 da., $5\frac{1}{2}$%.

72. Squares and Square Roots. By examining the slide rule it will be seen that the numbers on *A* are the squares of the numbers just below them on *D*. Hence, to find the square of any number, place the hair line of the runner over the number on *D* and read the square under the hair line on *A*.

Conversely, to find the square root of any number, place the hair line of the runner over the number on *A* and read the square root under the hair line on *D*.

The slide rule can also be conveniently used for the purpose of evaluating such expressions as a^2b. For example, to find the value of $8^2 \times 5$ we have the following arrangement:

A		Read 320, answer
B	Set 1	Over 5
C		
D	Over 8	

That is, set 1 (*B*) over 8 (*D*), and over 5 (*B*) read the answer, 320 (*A*). In this way we can easily find the value of πr^2.

We can also find the value of an expression in the form of $\sqrt{a/b}$. Thus, to find the value of $\sqrt{\frac{3}{4}}$ we have the following:

A	Under 3	
B	Set 4	Under 1
C		
D		Read 0.866, answer

That is, set 4 (*B*) under 3 (*A*), and under 1 (*B*) read the answer, 0.866 (*D*). In this way we can easily find the value of $\sqrt{a/\pi}$; that is, given the area of a circle we can find the radius.

Exercise 29. Squares and Square Roots

Find the square of each of the following:

1. 3.
2. 7.
3. 9.
4. 11.
5. 15.7.
6. 16.8.
7. 37.5.
8. 4.28.
9. 71.6.
10. 3.14.

Find the square root of each of the following:

11. 4.
12. 9.
13. 16.
14. 49.
15. 64.
16. 81.
17. 144.
18. 1.44.
19. 14.4.
20. 3.81.

21. Find the value of πr^2 when $r = 18.2$.

Notice that the mark for π is given on the common slide rule.

22. Find the area of a circle whose radius is 1.7 in.

23. Find the radius that must be used to describe a circle which shall have an area of 17 sq. in.

Find the value of each of the following:

24. 5.1×17.1^2.
25. 2.31×1.8^2.
26. 42×7.5^2.
27. 6.2×13^2.
28. 8×1.75^2.
29. 0.6×1.2^2.

30. What is the diameter of a cylindric iron rod whose cross-section area is 7.2 sq. in.?

31. What is the cross-section area of a cylindric iron rod whose diameter is 2.8 in.?

Find the value of each of the following:

32. $\pi \times 2.8^2$.
33. $\pi \times 7.4^2$.
34. $\pi \times 82^2$.
35. $\pi \times 720^2$.
36. $\pi \times 0.7^2$.
37. $\pi \times 0.91^2$.

Find the radii of the circles whose areas are as follows:

38. 276 sq. in.
39. 785 sq. ft.
40. 500 sq. yd.

73. Cubes and Cube Roots. We have learned how to find the value of a^2b. In this expression, if $b = a$ we have $a^2b = a^3$. That is, to find the cube of a number we simply multiply a^2 by a. Thus, to find the cube of 5 we have the following:

A		Read 125, answer
B	Set 1	Over 5
C		
D	Over 5	

That is, set 1 (*B*) over 5 (*D*), and over 5 (*B*) read the answer, 125 (*A*).

To find the cube root of a number we reverse the above process. We have given a^3, and desire to find a. In the example above, $a^3 = 125$. To find a we move the slide back and forth until the number on *B* under a^3 (*A*) is the same as that under 1 (*B*) or (*C*). In this case, when 5 (*B*) is under 125 (*A*), 5 (*D*) will be under 1 (*C*).

Exercise 30. Cubes and Cube Roots

Find the approximate value of each of the following:

1. 12^3.
2. 8^3.
3. 15^3.
4. 210^3.
5. 81^3.
6. 125^3.
7. 3.4^3.
8. 2.8^3.
9. 4.3^3.
10. 3.12^3.
11. 1.15^3.
12. 1.25^3.
13. 0.08^3.
14. 0.0015^3.
15. 0.00025^3.

16. Write the rule for determining the characteristic of the logarithm of the result in each case in Exs. 1–15.

Find the approximate value of each of the following:

17. $\sqrt[3]{1378}$.
18. $\sqrt[3]{125}$.
19. $\sqrt[3]{1450}$.
20. $\sqrt[3]{256}$.
21. $\sqrt[3]{85}$.
22. $\sqrt[3]{950}$.
23. $\sqrt[3]{24.6}$.
24. $\sqrt[3]{1.84}$.

Exercise 31. Review of Chapter III

1. In the table of logarithms on pages 231–248 find the mantissas of all the integers from 1 to 10 inclusive, multiply each by 5, and call each result inches. Draw a line *AB*, 5 in. long, and from *A* measure off distances on this line corresponding to the results obtained by the multiplication. Mark the points thus found 1, 2, 3, · · ·, 10, and compare the result with the *A* scale of a slide rule.

2. If required to complete for a 10-inch slide rule the results of Ex. 1, how would you proceed with the *A* scale?

3. With a slide rule estimate the approximate cost of excavating a cellar 35 ft. long, 30 ft. wide, and 5 ft. 6 in. deep at a cost of $1.35 per cubic yard.

4. Explain how to set the *A* and *B* scales to find the cost of cotton braid at the rate of 3 yd. for 25¢.

5. With a slide rule find the exact interest on $850 for 121 da. at 6%.

6. With a slide rule check the following items relating to eight workmen on the pay roll of a corporation which pays the employees of this class $26 per week of 44 hr.:

NUMBER	HOURS	WAGES		NUMBER	HOURS	WAGES	
247	40	23	63	269	44	26	
248	42	24	82	271	35	19	80
253	39	17	86	274	43	24	61
268	44	26		276	37	21	12

7. Explain how a slide rule could be constructed for chain discounts, using the *D* scale to find the net cost from the gross cost, the numbers on the *C* scale being replaced by net-cost-rate factors as multipliers.

Exercise 32. Review of Chapters I-III

1. Find the value of $25^0 + 25^{\frac{3}{2}} - 2^{-1} + 3\frac{1}{2} + 0^5$.

2. Multiply $x^{-2} + 2x^{-1} + 3$ by $2x^{-2} + 5x^{-1} + 6$.

Expand each of the following:

3. $(1+i)^5$. **5.** $(1+i)^7$. **7.** $(1+0.06)^2$. **9.** 1.06^4.
4. $(1+i)^6$. **6.** $(1+i)^8$. **8.** $(1+0.06)^3$. **10.** 1.06^5.

11. From the result of Ex. 10 find the amount of \$5200 at compound interest for 5 yr. at 6%.

12. Given that $x = 25^6 \div 3\sqrt{7}$, find the expression for $\log x$.

13. If $A = 1.035^7 \times \$7500$, find the value of A.

14. A certain depreciation-fund formula is

$$1 - x = \sqrt[n]{\frac{V_1}{V}}.$$

In this formula x is the constant rate of value to be written off each year, n the number of years, V_1 the salvage value, and V the original investment. If $n = 20$, $V = \$3500$, and $V_1 = \$400$, find the value of x.

15. If an egg crate holds 12 layers, each layer being 18 eggs long and 12 eggs wide, find by the slide rule the number of eggs in 5 such crates and then find the cost of the eggs at 35¢ per dozen.

16. With a slide rule find the value of the expression

$$\frac{35 \times 0.05 \times 125}{365 \times 7.45 \times 55{,}000}.$$

17. With a slide rule determine the characteristic of the logarithm of the result of the following computation:

$$973 \times 5487 \times 89.5 \times 5.3 \div 9.45;$$

then state the number of integral places in the result.

CHAPTER IV

SERIES

74. Series. A succession of numbers that is formed according to a definite law is called a *series*; the successive numbers are called the *terms* of the series.

For example, the numbers

$$2, \quad 5, \quad 8, \quad 11, \quad 14, \quad \cdots,$$

arranged in this order, form the *terms* of a series, the law being that each term after the first is found by adding 3 to the preceding term.

We also have a series in the case of the numbers

$$2, \quad 6, \quad 18, \quad 54, \quad 162, \quad \cdots,$$

the law being that each term after the first is found by multiplying the preceding term by 3.

The subject of series is very important in commercial algebra.

In arithmetic work the word *progression* is often used for series.

75. Finite and Infinite Series. A series in which the number of terms is limited is called a *finite series*. A series in which the number of terms is not limited is called an *infinite series*.

For example, the series 2, 6, 18, 54 is a finite series of four terms; but the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, continued forever, is an infinite series.

76. Arithmetic Series. A finite series in which each term after the first may be formed by adding a constant number to the preceding term is called an *arithmetic series*.

For example, 5, 12, 19, 26 is an arithmetic series of four terms, 7 being the constant number added to each of the first three terms to make the succeeding term. The series $5, 3, 1, -1, -3$ is an arithmetic series of five terms, the constant number added being -2.

77. Elements of an Arithmetic Series. An arithmetic series may be represented by

$$a, \quad a+d, \quad a+2d, \quad a+3d, \cdots,$$

in which a is the first term and d is the number added.

It is customary to speak of d as the *common difference*, and to represent the various elements as follows:

$a =$ first term, $d =$ common difference,

$l =$ last term, $n =$ number of terms,

and to let S stand for the sum of all the terms.

If d is positive, the series is an *increasing series*, as in the case of 2, 5, 8, 11, where $a = 2$, $d = 3$, $n = 4$, $l = 11$, and $S = 26$.

If d is negative, the series is a *decreasing series*, as in the case of 16, 12, 8, 4, 0, -4, in which $a = 16$, $d = -4$, $n = 6$, $l = -4$, and $S = 36$.

The terms between the first and last terms are called *arithmetic means*.

78. The *n*th Term, or *l*. Since each term of an arithmetic series, after the first, is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term. Hence the coefficient of d in the nth term is $(n-1)$. Calling the nth term l, we have

$$\boldsymbol{l = a + (n-1)d.}$$

Thus in the series 2, 5, 8, 11, we see that the last term is 11, and that $11 = 2 + (4-1)\cdot 3 = 2 + 3\cdot 3$.

79. Sum of the Terms. Indicating the sum of the terms by S, we have

$$\begin{aligned}
S &= a + (a+d) + (a+2d) + \cdots + (l-d) + l\\
\text{or}\quad S &= l + (l-d) + (l-2d) + \cdots + (a+d) + a\\
\hline
\therefore 2S &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)\\
&= n(a+l).
\end{aligned}$$

Dividing by 2, we have the formula for the sum:

$$\boldsymbol{S = \frac{1}{2}n(a+l).}$$

80. Problems in Arithmetic Series. If the values of any three of the letters in the formula of § 78 or of § 79 are known, the value of the other letter can be found.

1. Find the tenth term of the series \$1.05, \$1, \$0.95, ...

We have $a = \$1.05,\ d = -\$0.05,\ n = 10.$

Therefore
$$\begin{aligned} l &= a + (n-1)d \\ &= \$1.05 + 9 \times (-\$0.05) \\ &= \$0.60. \end{aligned}$$

In commercial work it is often desirable to preserve the dollar sign. We shall see this desirability when we study compound interest.

2. Find to 10 terms the sum of the series of Ex. 1.

Since we have found that $l = \$0.60$, we have
$$\begin{aligned} S &= \tfrac{1}{2}n(a+l) \\ &= 5(\$1.05 + \$0.60) \\ &= \$8.25. \end{aligned}$$

3. Write the arithmetic series of which the first term is 3, the last term is 21, and the sum is 120.

From the formula $S = \frac{1}{2}n(a+l)$

we have $120 = \frac{1}{2}n(3+21),$

and hence $n = 10.$

From the formula $l = a + (n-1)d$

we have $21 = 3 + (10-1)d,$

and hence $d = 2.$

Therefore the series is 3, 5, 7, 9, 11, 13, 15, 17, 19, 21.

4. Insert three arithmetic means between 2 and 14.

We have $a = 2,\ l = 14,$ and $n = 5.$

Since $l = a + (n-1)d,$

we have $14 = 2 + (5-1)d,$

and hence $d = 3.$

Therefore the series is 2, 5, 8, 11, 14, and the three means are 5, 8, 11.

Exercise 33. Arithmetic Series

Examples 1 to 8, oral

State which of the following series are arithmetic, stating the common difference in each series:

1. 5, 7, 9, 11, ...

2. 30, 25, 20, 15, ...

3. -6, -2, 2, 6, ...

4. 3, 6, 12, 24, ...

5. x, $5x$, $9x$, $13x$, ...

6. x, x^2, x^3, x^4, ...

7. 5, 1, -3, -7, ...

8. 3, $1\frac{1}{2}$, 0, $-1\frac{1}{2}$, ...

Given the following elements of an arithmetic series, write the first eight terms of each series:

9. $a = 3$, $d = 2$.

10. $a = 3x$, $d = x - 1$.

11. $a = -8$, $d = 3$.

12. $a = 4$, $d = -5$.

In the following series find the term specified in each case:

13. 7, 11, 15, ...; sixth term.

14. 3, -7, -17, ...; twentieth term.

15. \$1.05, \$1.10, \$1.15, ...; fifteenth and fiftieth terms.

16. \$10.40, \$10.30, \$10.20, ...; thirtieth term.

17. \$8.40, \$8.35, \$8.30, ...; twenty-fifth term.

18. 3, $3\frac{1}{4}$, $3\frac{1}{2}$, ...; sixteenth and thirtieth terms.

19. 3, 6, 9, ...; nth term and $(n+1)$th term.

20. \$45, \$44.50, \$44, ...; kth term.

In each of the following series find the sum of the number of terms specified:

21. 2, 4, 6, 8, ...; 60 terms.

22. \$1.05, \$1, \$0.95, ...; 21 terms.

23. -3, -5, -7, ...; 11 terms and 22 terms.

Insert the number of arithmetic means specified:

24. Two between 5 and 11. **26.** One between 6 and 16.

25. Five between 3 and 27. **27.** One between $3x$ and $15x$.

In each of the following series find the sum of the number of terms specified:

28. 3.4, 4.1, 4.8, $\cdots$; 18 terms and 20 terms.

29. $1+3x$, $1+5x$, $1+7x$, $\cdots$; 8 terms.

30. p, $3p$, $5p$, $\cdots$; 10 terms and n terms.

Given these elements, write the arithmetic series:

31. $a=5$, $l=38$, $S=258$. **32.** $a=3$, $l=-17$, $S=-77$.

33. If a man should dig a well 125 ft. deep, receiving 30¢ for the first foot and 3¢ more thereafter for each foot than for the preceding one, how much would he receive in all?

34. A bookkeeper receiving $960 a year with no prospect of an increase is offered another position at $800 for the first year and an annual increase of $50 a year for 10 yr. Find the difference in salary in 3 yr.; in 10 yr.

35. A factory pays each man 24¢ an hour during his first year, 26¢ an hour during his second year, and so on until he reaches a maximum of 36¢ an hour. During which year of service will he first receive the maximum salary?

36. In Ex. 35 what is the average wage per hour from the time a man begins to work in the factory until he reaches the period of maximum salary?

37. In Ex. 35, allowing 44 hr. to the week and 50 wk. of work to the year, find the total wages received by a workman who enters the factory at 24¢ an hour and remains through the year in which he receives 36¢ an hour.

81. Installment Payments with Interest. By one form of contract with houses doing an installment business the purchaser agrees to pay a specified sum for the article bought, paying in equal installments, with interest on all unpaid balances, the interest being paid with the last installment.

For example, a dealer sells a piano for \$500, receiving \$50 cash and the balance in installments of \$10 a month, with interest at 6% on all unpaid balances, the interest being paid with the last installment. Find the amount of the interest.

The balances and interest are evidently as follows:

UNPAID BALANCES	INTEREST FOR 1 MO. @ 6%
\$450	\$2.25
440	2.20
...	...
10	.05

In the interest payments we have an arithmetic series in which $a = \$0.05$, $d = \$0.05$, $n = 45$, and $l = \$2.25$. Hence

$$S = \tfrac{45}{2}(\$0.05 + \$2.25) = \$51.75.$$

Exercise 34. Installment Payments with Interest

1. A man bought a phonograph for \$175, paying \$25 cash and the balance in installments of \$5 a month, with interest at 6% on the unpaid balances. Find the amount of the interest.

2. A man bought a house for \$5000, paying \$1000 cash and the balance in installments of \$500 a year, with interest at 6% on the unpaid balances. Find the total amount paid.

3. An installment house offers a set of furniture for \$10 cash and \$5 a month for 24 mo., with 1% a month on the unpaid balances. Find the total cost.

4. The Aeolus Company sold a \$2000 piano for \$250 cash and the balance in installments of \$25 a month, with interest at 6% on the unpaid balances. Find the total cost.

82. Geometric Series. A finite series in which each succeeding term may be formed by multiplying the preceding term by a constant multiplier is called a *geometric series.*

The constant multiplier is called the *ratio.*

The first term is designated by a, the nth or last term by l, the ratio by r, the number of terms by n, and the sum by S.

Special examples of a geometric series are

$$2,\quad 4,\quad 8,\quad 16,\quad 32,\quad 64,\quad 128,$$

and

$$729,\quad 243,\quad 81,\quad 27,\quad 9,\quad 3,\quad 1,\quad \tfrac{1}{3},$$

and the general form is

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad ar^4,\quad \cdots.$$

The terms between the first and last terms are called *geometric means.*

83. The *n*th Term, or *l*. Since each term of a geometric series, after the first, is obtained by multiplying the preceding term by r, the exponent of r is always one less than the number of the term, so that the second term is ar, the third ar^2, the tenth ar^9, and the nth, or last term, ar^{n-1}. Hence

$$\boldsymbol{l = ar^{n-1}}.$$

Thus in the series 5, 15, 45, 135, we see that $135 = 5 \cdot 3^3 = 5 \cdot 27$.

84. Sum of the Terms. To find the sum we have

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiplying by r, $\quad rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$

Subtracting, $\quad rS - S = ar^n - a,$

or $\quad (r-1)S = ar^n - a.$

Hence $$\boldsymbol{S = \frac{ar^n - a}{r-1}}.$$

Since $l = ar^{n-1}$, it follows that $lr = ar^n$, and therefore by substituting in the above equation we have

$$\boldsymbol{S = \frac{lr - a}{r-1}}.$$

We therefore have two convenient formulas for S.

85. Problems in Geometric Series. If the values of any three of the letters in the formula of § 83 or of § 84 are known, the value of the other letter can be found.

1. Find the sum of 11 terms of the series 3, 6, 12, ...

We have $a = 3$, $r = 2$, $n = 11$. Therefore

$$S = \frac{ar^n - a}{r - 1} = \frac{3 \times 2^{11} - 3}{2 - 1} = 6141.$$

2. In a geometric series it is given that $a = 2$, $l = 486$, and $r = 3$. Find n and S.

Since $$l = ar^{n-1},$$

we have $$486 = 2 \times 3^{n-1}.$$

Therefore $$(n-1)\log 3 = \log 243,$$

$$(n-1) \times 0.477121 = 2.385606,$$

and $$n = 6.$$

Also $$S = \frac{lr - a}{r - 1} = \frac{486 \times 3 - 2}{3 - 1} = \frac{1456}{2} = 728.$$

Exercise 35. Geometric Series

In these series find the sums specified:

1. 1, 2, 4, ...; eighth term.

2. 16, 8, 4, ...; fifth term.

3. 3, 6, 12, ...; kth term.

4. 2, 4, 8, ...; nth term.

5. $1 + i$, $(1 + i)^2$, $(1 + i)^3$, ...; kth term and $(k + 1)$th term.

In these series find the sums specified:

6. 3, 6, 12, ...; 10 terms.

7. 5, 10, 20, ...; 10 terms.

8. 1.06, 1.06^2, 1.06^3, ...; 8 terms.

9. $1 + r$, $(1 + r)^2$, $(1 + r)^3$, ...; 15 terms and n terms.

10. x^2, x^4, x^6, ...; 12 terms, n terms, and $n + 1$ terms.

11. If $a = 1$, $r = 2$, and $l = 1024$, find n and S.

86. Infinite Geometric Series. When r is a proper fraction the successive terms become numerically smaller and smaller. By taking n large enough we can therefore make the nth term as small as we please, bringing it nearer and nearer to zero.

Thus in the series $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$, the terms are getting smaller and smaller. If we take $n = 14$, we shall have $l = \frac{1}{2048}$, a small fraction; and if we take $n = 21$, we shall have $l = \frac{1}{262144}$, a very small fraction.

Since $r < 1$, we may avoid negative terms by writing $S = \dfrac{ar^n - a}{r-1}$ in the form $\dfrac{a - ar^n}{1-r}$; and by taking n large enough we can make ar^n as near zero as we please, and can make S approach as near as we please to the *limit* $\dfrac{a-0}{1-r}$, or $\dfrac{a}{1-r}$.

For convenience it is usually said that

$$S = \frac{a}{1-r},$$

where r is a proper fraction and n is infinite.

87. Illustrative Examples. 1. Find the sum of the infinite series $10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$.

Since $a = 10$, $r = \dfrac{1}{2}$; and since $S = \dfrac{a}{1-r}$, we have $S = \dfrac{10}{1-\frac{1}{2}} = 20$. That is, the further we go in summing the series, the nearer the sum approaches 20.

2. Find the sum of the infinite series $9, -3, 1, -\frac{1}{3}, \cdots$.

Here $a = 9$, and $r = -\frac{1}{3}$.

Therefore $S = \dfrac{a}{1-r} = \dfrac{9}{1-(-\frac{1}{3})} = \dfrac{9}{\frac{4}{3}} = \dfrac{27}{4} = 6\frac{3}{4}$.

3. Find the sum of the infinite series $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots$.

Here $a = \frac{1}{10}$, and $r = \frac{1}{10}$.

Therefore $S = \dfrac{a}{1-r} = \dfrac{0.1}{1-0.1} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$.

That is, the limit of the decimal fraction $0.111\cdots$ is $\frac{1}{9}$.

4. The present value of \$800 due in 1 yr., money being worth 5%, is \$800 ÷ 1.05; the present value of \$800 due in 2 yr. is $\$800 \div 1.05^2$; in 3 yr., $\$800 \div 1.05^3$, and so on. By the use of infinite series find the present value of \$800 payable at the end of each year, the series of payments continuing indefinitely and money being worth 5%.

We have the infinite geometric series $\frac{800}{1.05}, \frac{800}{1.05^2}, \frac{800}{1.05^3}, \cdots,$ in which $a = \frac{800}{1.05}$ and $r = \frac{1}{1.05}$.

From § 86,
$$S = \frac{\frac{800}{1.05}}{1 - \frac{1}{1.05}} = \frac{800}{1.05} \times \frac{1.05}{0.05} = \frac{800}{0.05} = 16{,}000.$$

That is, the present value is \$16,000.

Exercise 36. Infinite Geometric Series

Find the sum of each of the following infinite series:

1. $15, 5, \frac{5}{3}, \cdots$.
2. $32, 16, 8, \cdots$.
3. $10, 3\frac{1}{3}, 1\frac{1}{9}, \cdots$.
4. $\frac{21}{100}, \frac{21}{10000}, \frac{21}{1000000}, \cdots$.
5. $0.15, 0.0015, 0.000015, \cdots$.
6. $0.07, 0.0007, 0.000007, \cdots$.
7. Find the value of $0.5 + 0.04 + 0.004 + 0.0004 + \cdots$.

Find the value of each of the following:

8. $0.373737\cdots$.
9. $0.0444\cdots$.
10. $0.3171717\cdots$.

Find the sum of each of the following infinite series:

11. $\frac{300}{1.04}, \frac{300}{1.04^2}, \frac{300}{1.04^3}, \cdots$.
12. $\frac{250}{1.03}, \frac{250}{1.03^2}, \frac{250}{1.03^3}, \cdots$.
13. $\frac{1200}{1.05}, \frac{1200}{1.05^2}, \frac{1200}{1.05^3}, \cdots$.
14. $\frac{2000}{1.06}, \frac{2000}{1.06^2}, \frac{2000}{1.06^3}, \cdots$.

Exercise 37. Review of Chapter IV

1. Find the sum of the integers from 1 to 100 inclusive.

2. Find the sum of the even numbers from 224 to 336.

3. Find the sum of the odd numbers from 135 to 215.

4. A man borrowed $6000 and repaid it in ten equal annual payments at the end of the respective years, with interest at 6%. Find the total amount paid.

5. Find the sum of the series $1.05^{10} \times 500$, $1.05^{9} \times 500$, $1.05^{8} \times 500, \cdots, 1.05 \times 500, 500$.

Find the value of each of the following:

6. $300 + 1.06 \times 300 + 1.06^2 \times 300 + \cdots + 1.06^{12} \times 300$.

7. $1.04^8 \times 900 + 1.04^7 \times 900 + 1.04^6 \times 900 + \cdots + 900$.

8. $1.05^3 \times 700 + 1.05^2 \times 700 + 1.05 \times 700 + 700$.

9. If you were to cut off half of a 40-inch line, then half of the remainder, then half of the next remainder, and so on, after how many operations would the line be $\frac{5}{8}$ in. long?

10. If you pay $\frac{1}{4}$ of a $16,000 debt this year, $\frac{1}{4}$ of the remainder next year, $\frac{1}{4}$ of the next remainder the third year, and so on, after how many years would you owe $5062.50?

11. If a rubber ball let fall from a point 6 ft. above the floor could always rebound half the distance it falls to a floor, and continue thus to rebound however short the fall, what would be the distance it would traverse if the fall and the rebound continued indefinitely?

12. A certain bacillus multiplies by self-bisection. If there were 2500 bacilli in a glass jar and if self-bisection averaged once in every 40 hr., and none died, how many separate bacilli would there be in the jar at the end of 40 da.?

Use logarithms, expressing the result by a number of four figures multiplied by some power of 10.

Exercise 38. Review of Chapters I–IV

1. By the aid of logarithms, find the product of $1.06^{\frac{1}{2}}$ and $1.06^{\frac{3}{2}} \times \$800$.

2. Find the value of the expression $5^0 + 4\frac{1}{2} + 4^{\frac{1}{2}} - 2^{-1} + 0^{20}$.

3. If 4 were the base of a system of logarithms, what would be the value of log 1? of log 16? of log 64?

4. By the aid of logarithms, find the value of

$$365{,}500 \times 48{,}970 \times 75.46 \div (45 \times \sqrt[3]{7500}).$$

5. Find the value of $1 \div 1.025^{16}$ to four significant figures.

In such a case 0 is considered a significant figure when it occurs between two other figures, expressed or understood. For example, 0 is not a significant figure in 0.075, but it is a significant figure the second time that it occurs in 0.308, and the second and third times that it occurs in 0.43007.

6. Find the value of the expression $\frac{5}{0.0225}\left(1 - \frac{1}{1.0225^8}\right)$, using logarithms to find the value of the second fraction.

7. Make a rough drawing of a slide rule set to check a pay roll, the rate of wages being 37¢ per hour. Show only the marks brought into coincidence to set the rule and the marks that will indicate the wages due for 42 hr. of work.

With a slide rule determine the correction to be made, if any, in each of the following statements:

8. $17 \times \$3.40 = \$56.80$. **9.** $\$94.60 \div 28 = \$3.38$.

10. $13.5 \times 9.7 = 130.9$ approximately.

11. Find the value of the decimal $0.3151515 \cdots$.

12. Using the formula for S in an arithmetic series, find the value of $\$2.25 + \$2.20 + \$2.15 + \cdots + \$1.15$.

13. Given that $a = 2$, $l = 57$, and $d = 5$ in an arithmetic series, find the value of n.

CHAPTER V

COMPOUND INTEREST

88. Compound Interest. If simple interest is not paid promptly at the end of the year, or of some other agreed interval, the interest then due does not itself draw interest from that time to the time of settlement. Penalties are usually provided, however, in the contract, which secure the prompt payment of the interest when due.

When interest is paid promptly, it may be put at interest, and thus in effect be added to the principal. Many large corporations, such as life-insurance companies, are receiving funds constantly and are promptly reinvesting them. Even when the interest due on a note at stated intervals is not paid, a new note may be made and the interest thus due may then be added to the principal, the interest for the next interest period being then calculated on the sum.

That form of interest in which the interest is added to the principal at stated intervals is called *compound interest*, a subject already somewhat familiar to the student. In the case of compound interest the interest is said to be *converted* into principal and the principal is said to *accumulate.*

The time elapsing between any two successive periods when the interest is added to the principal is called the *conversion period.*

For example, if a man deposits \$100 on Jan. 1 in a savings bank that pays 4% per annum, adding interest to the principal on Jan. 1 and July 1, the amount to his credit on July 1 is 1.02 × \$100, or \$102. In this case the conversion period is 6 mo. and the principal has been converted into \$102, a sum which now bears interest as before.

89. Compound Amount. The amount of principal and compound interest paid in settlement of an obligation is called the *compound amount.* The compound interest is the difference between the sum borrowed and the compound amount.

For example, if a man deposits $500 in a savings bank that pays 4%, the interest being paid semiannually, at the end of 6 mo. there is added to the principal 2% of $500, or $10. During the second 6 mo. the bank allows the depositor 2% of $510, or $10.20 interest. At the end of the year he may withdraw $520.20. This is the compound amount, and $20.20 is the compound interest. In the year there have been two conversion intervals. Since adding 2% of a principal to itself is the same as multiplying it by 1.02, it is evident that the amount is equal to $1.02 \times 1.02 \times \$500$, or $1.02^2 \times \$500$.

To avoid confusion of symbols for ratio in geometric series and interest rate, we shall now use r for the former and i for the latter. Let n be the number of conversion intervals during which the money is invested. Let the principal be $1, which we shall call simply the abstract number 1.

At the end of the first interval the amount is $(1+i)$.
During the second interval the interest is $(1+i)i$.
The amount at the end of the second interval is $(1+i)+(1+i)i$,
or $(1+i)^2$.
During the third interval the interest is $(1+i)^2 i$.
The amount at the end of the third interval is $(1+i)^3$.
Similarly, at the end of the fourth interval the amount is $(1+i)^4$,
and so on, and at the end of the nth interval the amount is $(1+i)^n$.
If we let A stand for the compound amount of 1 for n years or intervals at the rate i, we have

$$A=(1+i)^n.$$

The result found in this formula is the compound amount of 1 for n intervals at the rate i per interval, the compound interest being $(1+i)^n-1$.

90. Compound Amount for any Principal. It is evident that if the principal in § 89 had been P dollars instead of \$1, the result at the end of each interval would have been P times as great. Hence, if we represent the compound amount by A, we have

$$A = P(1+i)^n.$$

In theoretical calculations we assume that this formula holds good whether n is integral or fractional.

If $P = \$800$ and $i = 0.06$, we have

$$A = \$800 \times 1.06^n.$$

In such a substitution it is convenient to write the multiplier second, as it appears in the formula, and to read the symbol × "multiplied by." It is frequently convenient to keep the dollar sign in the substitution, and the student should use his judgment in the matter when the case arises.

91. Accumulation Factor. In the formula of § 90 the factor $(1+i)^n$ is called the *accumulation factor* or the *compound amount of 1.*

Both of these terms are used by investment computers.

The finding of the accumulation factor is the most difficult part of computing compound amounts. For example, in the case of $\$800 \times 1.06^3$, to find that $1.06^3 = 1.191016$ requires considerable time. This time is saved by the use of a table of accumulation factors such as the one given on pages 212–215.

Since a table is not always available, and since no table gives all possible rates, it is important to be able to find accumulation factors in other ways, and two methods will now be considered.

92. Approximations of $(1+i)^n$ by the Binomial Theorem. If n is small, it is often easier to develop $(1+i)^n$ by the Binomial Theorem than to consult a table.

For example, since

$$(1+i)^3 = 1 + 3i + 3i^2 + i^3,$$

we have

$$1.04^3 = 1 + 3 \times 0.04 + 3 \times 0.04^2 + 0.04^3$$
$$= 1.124864.$$

This result is exact, all terms of the expansion having been used.

93. Binomial Coefficients. For practical work as in the example in § 92 it is convenient to know the binomial coefficients for powers of $1+i$. As we have learned, $(1+i)^0=1$; expanding $(1+i)^n$ for $n=1, 2, 3, \cdots$, and taking the coefficients, we have the following scheme:

Powers of $(1+i)$	*Coefficients*
$(1+i)^0=1$	1
$(1+i)^1=1+i$	1 1
$(1+i)^2=1+2i+i^2$	1 2 1
$(1+i)^3=1+3i+3i^2+i^3$	1 3 3 1
$(1+i)^4=1+4i+6i^2+4i^3+i^4$	1 4 6 4 1

Any coefficient in the triangle of coefficients may be found by adding the coefficient above it to the one to the left of that coefficient.

Find an approximate value of 1.04^5, correct to 5 decimals.

Since the coefficients are 1, 5, 10, 10, 5, 1, we have $(1+0.04)^5=1+5\times 0.04+10\times 0.04^2+10\times 0.04^3+5\times 0.04^4+0.04^5$. These values are here given except for the last term, which does not affect the first five places.

$$\begin{array}{l} 1. \\ .20 \\ .0160 \\ .000640 \\ \underline{.00001280} \\ 1.21665 \end{array}$$

Exercise 39. Finding the Accumulation Factor

Examples 1 to 8, oral

State the value of each of the following:

1. 1.02^2.
2. 1.03^2.
3. 1.04^2.
4. 1.05^2.
5. 1.06^2.
6. 1.02^3.
7. 1.025^2.
8. 1.03^3.
9. Find the value of 1.04^8, correct to five decimals.
10. Find the compound amount of \$450 for 6 yr. at 5%.
11. Find the compound amount of \$250 for 10 yr. at 4%.

94. Compound Amount by Logarithms. It sometimes happens that the value of i or of n falls outside the limits of the table of compound amounts of 1. In such cases the value of $(1+i)^n$ and hence of A may be found by logarithms.

To secure accuracy to six figures a full six-place table is necessary.

95. Illustrative Examples. **1.** Find the compound amount of \$8560 for 7 yr. at $4\frac{1}{4}\%$, payable annually.

Since $$A = P(1+i)^n,$$

we have $$\log A = \log P + n \log (1+i).$$

Therefore $$\log A = \log 8560 + 7 \log 1.0425.$$

We have
$$\begin{aligned} \log 8560 &= 3.932474 \\ 7 \log 1.0425 &= \underline{0.126532} \end{aligned}$$

and so $$\log A = 4.059006$$

whence $$A = 11{,}455.3.$$

Hence the compound amount is \$11,455.30.

2. Find the compound interest on \$800 for 3 yr. 6 mo. at 6%, payable annually.

We have $$A = 800 \times 1.06^{\frac{7}{2}}.$$

Therefore $$\log A = \log 800 + \tfrac{7}{2} \log 1.06 = 2.903090 + 0.088571.$$

Therefore $$\log A = 2.991661, \text{ and } A = 980.98.$$

Therefore the compound interest is \$980.98 – \$800, or \$180.98.

Exercise 40. Compound Amount by Logarithms

Interest being payable annually, in each of the following cases find the compound amount and the compound interest:

1. \$1500, 8 yr., 5%.

2. \$300, 6 yr., $3\frac{1}{2}\%$.

3. \$900, 5 yr., 6%.

4. \$2500, 4 yr., 4%.

5. \$3000, 5 yr. 6 mo., 4%.

6. \$4000, 3 yr. 5 mo., 4.1%.

7. \$5000, 4 yr. 4 mo., 5.4%.

8. \$8000, 5 yr. 2 mo., $4\frac{1}{4}\%$.

96. Accumulation Factor by Multiplying. We may find the value of $(1+i)^5$ by multiplying $(1+i)^3$ by $(1+i)^2$, and similarly for other powers of $1+i$.

For example, given that $1.04^2 = 1.0816$ and $1.04^3 = 1.124864$, these having been found by the Binomial Theorem, we may find the compound amount of $500 for 5 yr. at 4 % thus:

$$1.04^5 = 1.04^2 \times 1.04^3 = 1.0816 \times 1.124864 = 1.216652+.$$

Hence $A = 1.216652 \times \$500 = \$608.33.$

97. Converse Cases. To find n or i from the formula $A = P(1+i)^n$ we proceed as follows:

$$\log A = \log P + n \log(1+i);$$

whence $$n = \frac{\log A - \log P}{\log(1+i)},$$

and $$\log(1+i) = \frac{\log A - \log P}{n}.$$

98. Illustrative Examples. 1. In what length of time will a deposit of $600 amount to $1000 in a savings bank that pays 2% every 6 mo.?

If n is the number of half-year periods, we have

$$1000 = 600 \times 1.02^n;$$

whence $$\log 1000 = \log 600 + n \log 1.02.$$

Solving, $$n = \frac{\log 1000 - \log 600}{\log 1.02} = \frac{3 - 2.778151}{0.008600} = 25.7+.$$

Following bank customs, we have 26 intervals, or 13 yr.

2. At what rate will $1000 amount to $1410.60 in 10 yr., at compound interest?

From § 97, $$\log(1+i) = \frac{\log 1410.60 - \log 1000}{10} = \frac{3.149404 - 3}{10}$$
$$= 0.014940.$$

Therefore $1 + i = \text{antilog } 0.014940 = 1.035.$

That is, $i = 3\frac{1}{2}\%.$

Exercise 41. Converse Cases of Compound Interest

1. How long will it take \$800 to amount to \$1000 at 5%, the interest being compounded annually?

2. How long will it take \$2500 to amount to \$3500 at 4%, the interest being compounded semiannually?

3. How long will it take \$2500 to amount to \$3500 at 4%, the interest being compounded quarterly?

4. A corporation began business with a plant and equipment valued at \$250,000. In 8 yr. it increased this valuation to \$350,000, besides paying dividends. Assuming that the rate of increase was uniform, find the annual rate.

5. The available coal supply of the United States in 1912 was estimated to be 3,200,000,000,000 T. It was also estimated that 440,000,000 T. were mined in that year and that there would be an average annual increase thereafter of 6% over each preceding year. At this rate, in how many years after 1912 will the coal supply be exhausted?

6. A corporation having a plant valued at \$2,000,000 wishes to double its valuation in 10 yr., increasing the investment annually by the same per cent of itself. Find the constant rate of increase.

7. A man leaves \$10,000 to a college on condition that the interest shall be added to the principal annually until the sum amounts to \$20,000, when the income shall become available. If the investment can be made to yield 5% compound interest, how long before the income will become available?

8. A man decides to divide \$20,000 between his two sons in such a way that the shares which the sons receive at the age of 21 yr., accumulated at 4% compound interest, shall be equal. If the sons are 13 yr. and 17 yr. old respectively, how much shall he give to each?

99. Nominal and Effective Rates of Interest. When interest is accumulated more often than once a year, it is customary to state the rate as a rate per annum, payable quarterly, semiannually, and so on. Thus a bond bearing interest at 6% per annum, payable January and July, means that 3% of the face of the bond is payable each half year.

The stated rate per annum is called the *nominal rate.* The rate actually realized per annum, due to conversion more often than once a year, is called the *effective rate.*

In the case of interest at 6%, convertible semiannually, s, the amount of 1, is found as follows:

$$s = 1.03^2 = 1.0609.$$

The holder of the security realizes $1.0609 - 1$, or 0.0609 for the year, 0.0609 being the effective rate and 0.06 the nominal rate.

100. Effective-Rate Formula. Let N represent the nominal rate of interest, E the effective rate, and m the number of conversion intervals per year. The rate per conversion interval is evidently N/m and the compound amount of 1 at the end of the year is therefore $(1+N/m)^m$. The interest realized on 1, or the effective rate, is therefore expressed by the formula

$$\boldsymbol{E=\left(1+\frac{N}{m}\right)^m-1.}$$

101. Nominal-Rate Formula. When the effective rate, E, and the number of conversion intervals are given, it is possible to find the corresponding nominal rate by solving for N the equation of § 100.

Transposing 1, $$1+E=\left(1+\frac{N}{m}\right)^m.$$

Taking the mth root, $$(1+E)^{\frac{1}{m}}=1+\frac{N}{m}.$$

Hence $$m(1+E)^{\frac{1}{m}}-m=N,$$

or $$N=m[(1+E)^{\frac{1}{m}}-1].$$

102. Illustrative Examples. **1.** Find the effective rate per annum if the nominal rate is 6%, payable quarterly.

By § 100, $$E = \left(1 + \frac{N}{m}\right)^m - 1.$$

Here $m = 4$, $N = 0.06$. Hence $E = \left(1 + \frac{0.06}{4}\right)^4 - 1 = (1 + 0.015)^4 - 1.$

Expanding by the Binomial Theorem we have

$(1 + 0.015)^4 = 1 + 4 \times 0.015 + 6 \times 0.015^2 = 1.0614$, approximately.

Hence $$E = 1.0614 - 1 = 0.0614.$$

2. What nominal rate will realize an annual return of 6% on an investment, the payments being made quarterly?

By § 101, $$N = 4\,(1.06^{\frac{1}{4}} - 1).$$

It now becomes necessary to use the Binomial Theorem (§ 16) for a fractional exponent, and we simply substitute $\frac{1}{4}$ for n in the expansion of $(a + b)^n$. We then have

$$(1 + 0.06)^{\frac{1}{4}} = 1 + \frac{1}{4} \times 0.06 + \frac{\frac{1}{4}(\frac{1}{4} - 1)}{2} \times 0.06^2 + \frac{\frac{1}{4}(\frac{1}{4} - 1)(\frac{1}{4} - 2)}{6} \times 0.06^3 + \cdots,$$

or $$1.06^{\frac{1}{4}} = 1 + 0.015 - 0.0003375 + 0.0000118 - 0.0000005 = 1.014674, \text{ approximately.}$$

Therefore $N = 4\,(1.014674 - 1) = 0.0587$, approximately.

Exercise 42. Nominal and Effective Rates

1. Convert a nominal rate of 4%, payable semiannually, into the equivalent effective rate.

2. If money is worth 5% nominal, payable semiannually, what equivalent annual rate should be received?

3. If the effective rate is 8%, find the equivalent nominal rate when money is converted four times a year.

4. Find the effective rate on money invested in a company paying monthly dividends of 2% reinvested in the company.

103. Discount. The word *discount* has various meanings. To the merchant buying or selling goods it means a reduction of a given per cent from the face of a bill or from the selling price of an article. In general, discount is a deduction from the amount of a debt, and when made for the payment of money before it is due the word is commonly used alone. Bank discount and trade discount have special meanings.

104. Present Value. When a debt is due at some future time, such a sum as, put at interest at the current rate, will amount to the full debt when that debt is due is called the *present value* of the debt.

Thus, if I owe \$11,236, due in 2 yr., and money is worth 6% compound interest, \$10,000 paid now will yield enough to pay the debt when due.

For

$$\begin{aligned} A &= P(1+i)^n \\ &= 10{,}000 \times 1.06^2 \\ &= 11{,}236. \end{aligned}$$

Hence the discount is \$1236, and the present value, represented in the formula by P, is \$10,000.

105. Present Value at Simple Interest. In the case of simple interest we have

$$a = p(1 + it);$$

whence

$$p = \frac{a}{1 + it}.$$

Therefore, *to find the present value of a debt at simple interest, divide the amount of the debt by the amount of 1 for the given rate and time.*

In the case of the present value of \$1260 due in 1 yr. 6 mo., money being worth 6% simple interest, we have

$$p = \frac{\$1260}{1.09} = \$1155.96.$$

The difference between the present value at simple interest and the face of the debt is sometimes called *true discount.* In this example the true discount is \$1260 − \$1155.96, or \$104.04.

Exercise 43. Present Value at Simple Interest

1. A gas company offers to sell a gas range for \$5 down, \$5 at the end of 1 mo., and \$5 at the end of 2 mo. Money being worth 6%, simple interest, find the equivalent cash price.

2. If a gas range can be purchased for \$5 down and \$3 a month for 5 mo., and money is worth 8% per annum to the firm that sells the range, find the equivalent cash price.

3. Find the difference between the bank discount and the true discount on a note for \$1800 due in 60 da., money being worth 6%.

Find the present value, at simple interest, of the following:

4. \$8000 due in 4 mo. at 6%; at 5%.

5. \$9500 due in 3 mo. at 5%; at 6%; at 4%.

6. \$12,500 due in 6 mo. at 4%; at $5\frac{1}{2}$%; at 6%.

7. A man presented to his bank for discount a note which would mature 33 da. later, the face being \$350 and the note bearing no interest. If the bank discounted the note at 6%, how much did the man realize upon it?

8. In Ex. 7 how much should the man have realized in order that the discount taken by the bank would represent 6% interest for 33 da. on the money advanced?

Ex. 7 represents the actual business custom; Ex. 8 represents what would be a more ideal method from the standpoint of the man who presents the note for discount.

9. What sum of money set aside now will amount to \$2500 in 4 yr. at 5% simple interest?

10. Find the present value of \$1000 due in 36 da., money being worth 5% per annum.

11. Find the present value of and the true discount on \$250 due in 60 da., money being worth 6% per annum.

106. Present Value at Compound Interest. When the interest on invested funds is paid promptly, money is in effect worth compound interest, for it usually can be reinvested. In most practical calculations relating to large investments present value is estimated on a compound-interest basis. The problem of finding present value, therefore, consists in finding P from the formula $A = P(1+i)^n$. Solving, we have

$$P = \frac{A}{(1+i)^n} = A \times \frac{1}{(1+i)^n},$$

the second form being the one actually used, the factor $1/(1+i)^n$ being given in the tables. Therefore,

To find the present value of a debt on a compound-interest basis, divide the amount of the debt by the compound amount of 1 for the time and rate, or multiply the amount of the debt by the present value of 1 for the time and rate.

Evidently $1/(1+i)^n$ represents the present value of 1 due in n years at the rate i; for we have $1 = P(1+i)^n$, so that $P = 1/(1+i)^n$, a fraction usually represented by v^n. Thus we have $v^3 = 1/(1+i)^3$, and at 4% we have $v^3 = 1/1.04^3$. A table of values of v^n is given on page 216. Beyond the limits of the table, logarithms may be used.

107. Illustrative Examples. 1. Find the present value of \$5000 due in 17 yr., money being worth 4%.

From the table on page 218, v^{17} at 4% is equal to 0.5133732. Hence

$$P = 0.5133732 \times \$5000 = \$2566.87.$$

2. Find the present value of \$2000 due in 5 yr., money being worth $4\frac{1}{4}\%$.

Since the table in this book does not give present values at $4\frac{1}{4}\%$, we use logarithms.

Since $\log P = \log A - n \log(1+i) = \log 2000 - 5 \log 1.0425$
$= 3.301030 - 0.090380 = 3.210650,$

we have $P = \text{antilog } 3.210650 = 1624.24.$

Hence the present value is \$1624.24.

108. Present Value of an Interest-Bearing Debt. If the face of an interest-bearing debt is A' and the rate of interest is k, the amount due at the end of n years is given by the formula

$$A = A'(1+k)^n.$$

Substituting for A in the formula $P = A/(1+i)^n$, we have

$$P = \frac{A'(1+k)^n}{(1+i)^n}.$$

For example, if a debt of \$850 bearing 5% interest is due in 4 yr., money being worth 4% at current rates, we have

$$P = \frac{\$850 \times 1.05^4}{1.04^4} = \$850 \times 1.2155063 \times 0.8548042 = \$883.17.$$

Exercise 44. Present Value at Compound Interest

1. Find the present value of \$480 due in 2 yr. without interest, money being worth 5%.

Here, as in all similar cases, this means that money is worth 5% per annum, payable annually. If paid more often, this will be stated in the problem. Thus, 8% payable quarterly means 2% every 3 mo.

2. In Ex. 1 suppose that the \$480 was due in 2 yr. with interest at 6%.

3. Find the present value of \$5000 due in 4 yr. with interest at 4%, money being worth 6%, payable semiannually.

4. A father wishes, at the birth of his son, to set aside such a sum as shall amount in 21 yr. to \$10,000. At 4%, compounded semiannually, how much must he set aside?

5. A man wishes to divide \$30,000 between his two sons, aged respectively 16 yr. and 19 yr., placing the share of each in the hands of a trustee to accumulate at 4% compound interest until each son reaches the age of 21 yr., when each shall receive the same amount. How much should he set aside for each?

109. Discount Rate and Interest Rate. Considering transactions involving short periods, we may define discount as the difference between a debt and its present value, and discount rate as the discount on a debt of 1 for 1 yr. Hence, representing the discount rate by d and the interest rate by i,

we have $$d = 1 - \frac{1}{1+i} = \frac{i}{1+i};$$

whence $$i = \frac{d}{1-d}.$$

These formulas give the relation between discount rate and interest rate for short periods. For example, to find at what rate a man could afford to borrow money to pay a bill on which he is allowed a discount of 2% for payment made 3 mo. before the bill is due, we have

$$i = \frac{d}{1-d} = \frac{0.02}{0.98} = 0.020408 \text{ for the 3 mo.}$$

The man could therefore afford to pay 4 × 0.020408, or 0.081632 per annum. If he can borrow money at 8% or less, it will pay him to do it.

Exercise 45. Discount Rate and Interest Rate

1. An invoice specifying "terms 2 mo. net" is subject to 3% discount for immediate payment. What is the highest rate of interest that the debtor can afford to pay for the money needed to avail himself of the discount?

2. If a merchant's money invested in business yields him 2% a month, what discount rate can he afford to grant for the immediate payment of a bill on which he quotes "net 30 da."?

3. A jobber received a bill for $500 on July 1 postdated Sept. 1, terms 3% 10 da., net 30 da., with 1% a month added from July 1 to the date of payment. If he can borrow money at 6%, how much will he gain by doing so and paying the bill on July 1?

4. Find the annual interest rate equivalent to a discount rate of 2% for 60 da.

5. Find the discount rate for 30 da. equivalent to an interest rate of 6% per annum.

6. The Reed Furniture Co. received a consignment from Owen & Shaw of Grand Rapids, the terms being stated on the invoice as 2/10, N/60, and the items being as follows:

35 F	350	Plush library chairs	@	$14.75
25 H	370	Reed rockers	@	3.10
50 F	45	Bedroom rockers	@	4.85
40 F	64	Roman chairs	@	7.85
35 H	245	Morris chairs	@	18.50
36 F	24	Dining tables	@	24.75

Find the gross amount of the invoice and the highest interest rate the Reed Furniture Co. can afford to pay for the money necessary to take advantage of the 2/10 discount.

Symbols like 35 F refer to the catalogue numbers.

7. The A. B. Hughes Co. estimates that the money put into their business yields a net return of $1\frac{1}{2}$% a month. Find the highest discount rate they can afford to allow for payment made 1 mo. before the net amount of a bill is due.

8. D. M. Long sold S. H. Bishop on account 30 da., 2% 10 da., a bill of lumber amounting to $12,500. Find the rate of interest Bishop can afford to pay for the money necessary to take advantage of the 2% discount on the tenth day.

9. In Ex. 8, if Bishop borrowed money at 6%, what rate of interest did he receive on it during the 20 da.?

10. Find the discount rate for 60 da. which is equivalent to an interest rate of 8% per annum.

11. Find the interest rate per annum which is equivalent to a discount rate of 1% for 60 da.

110. Simple and Compound Amounts. If n is the number of years that a loan runs and i is the rate of interest, the amounts at simple and compound interest respectively are given by the formulas

$$a = p(1 + in)$$

and $$A = P(1 + i)^n,$$

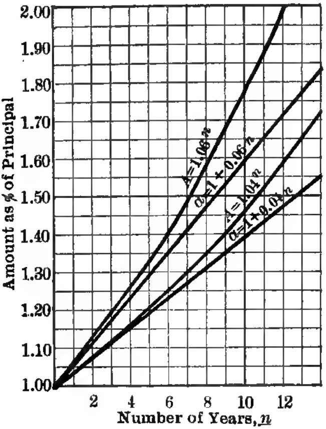
the interest being compounded annually.

If the principal is 1, these formulas become

$$a = 1 + in$$

and $$A = (1 + i)^n,$$

respectively, both these and the preceding forms being commonly used in commercial algebra.



Since we have three letters in each of the last two formulas, and since in our work in graphs we have used only two letters, we should give some definite value to one of the three letters if we wish to represent the formulas graphically in the usual manner. Letting $i = 0.04$, we see that the graph of the equation

$$a = 1 + 0.04\,n$$

becomes a straight line, while the graph of the equation

$$A = 1.04^n$$

is a curve which separates more and more from the straight line as n increases. This is seen better in the above figure for the case in which the rate is 6%. That is, the compound amount on any sum increases much more rapidly than the amount at simple interest for the same rate, the difference being greatly increased as the time increases.

Exercise 46. Simple and Compound Amounts

1. What is the difference, as shown by the graphs, between the amounts at simple and compound interest in 8 yr. at 6%?

2. By what per cent of the principal does the compound amount exceed the amount at simple interest in 10 yr. at 6%?

3. In what time will money double itself at 6% compound interest? at $5\frac{1}{2}$% compound interest?

4. In how many years will a principal increase 50% of itself at 6% compound interest?

5. In what time will a sum increase 60% of itself at 6% simple interest? at 6% compound interest? at 4% compound interest? at 3% compound interest?

6. How many per cent greater is the gain on a given principal in 10 yr. at 6% compound interest than at 4% compound interest?

7. How many per cent greater is the gain on a given principal in 8 yr. at 4% compound interest than at 4% simple interest?

8. Verify the result obtained from the graph for Ex. 7 by calculating $(1+0.04)^8$ by the Binomial Theorem, using four terms of the development.

9. In how many years do the amounts at compound and simple interest at 4% differ as much as the amounts at compound and simple interest at 6% differ in 6 yr.?

In Exs. 9–11 use graphs.

10. How many times as great is the difference in 12 yr. between the amounts at compound and simple interest at 6% as between the corresponding amounts at 4%?

11. In how many years do the differences of amount mentioned in Ex. 10 have the ratio of 2 : 1?

12. Copy the graph of page 94 and add similar lines for $a = 1 + 0.03\,n$ and $A = 1.03^n$.

13. Using the graph of Ex. 12, compare for divergence the 3% curves with the 4% and 6% curves in 8 yr., 10 yr., 12 yr.

14. Using the graph of Ex. 12, compare the divergence of the amounts at simple and compound interest at 3% with that at 6% for the ninth year.

15. What conclusions may be drawn from the work in § 110 and Exercise 46 as to the importance of the interest rate in the accumulation of wealth?

16. Draw the graph of $A = 1.08^n$ for all integral values of n from 2 to 12, locating points for every second year.

17. From some bank or trust company in your vicinity ascertain the current discount rate for short-term commercial paper. Then find the equivalent interest rate per annum earned by the bank or trust company.

18. Using the interest rate found in Ex. 17, find the compound amount of $250,000 in 20 yr.

19. A man left by will $25,000 to be invested, together with the interest earnings, for a period of 50 yr. Find the amount at $4\frac{1}{2}$%, compounded annually.

20. If money can be invested at 5%, compounded annually, how long will it take $1000 to amount to $1500?

21. From the formula $A = P(1 + i)^n$ deduce formulas for P, i, and n.

22. Construct a table showing the accumulation of the amount of $1000 at simple interest at 4% for 4 yr.

23. Construct a table showing the accumulation of the compound amount of $1000 invested at 4% annually for 4 yr.; invested at 5% annually for 5 yr.

Exercise 47. Review of Chapter V

1. Find the compound amount of \$5000 for 3 yr. at $3\frac{1}{4}\%$.

2. A wealthy man set aside \$500,000, creating a trust which is to administer this fund, investing it, and adding the interest accruals to the principal for 50 yr., when the amount is to be used to endow a hospital. If the average yield is $4\frac{1}{4}\%$ per annum, how much will be accumulated?

3. Assuming that the formula $A = P(1 + i)^n$ holds for all real positive values of i and n, find the compound amount of \$2500 in 5 yr. 4 mo. at $4\frac{1}{4}\%$.

4. What is the present value of a debt of \$3500 bearing interest at 4%, due in 4 yr., money being worth 5%?

5. The interest being paid promptly, so that the present value is calculated on a compound-interest basis, find the cash value of an article that can be bought for \$500 cash and \$500 at the end of each year for 5 yr., money being worth 4%.

6. What annual interest rate is equivalent to a discount rate of 3% for 3 mo.?

7. Using the Binomial Theorem, find the value of the accumulation factor 1.02^8, correct to 4 decimal places.

8. Find the value of 1.02^8 by logarithms.

9. In how many years will a deposit of \$400 amount to \$600 if left in a savings bank which pays 2% every 6 mo.?

10. At what rate must I invest money to increase my capital $\frac{1}{3}$ in 5 yr., assuming that I can reinvest the interest at the same rate?

11. A man buys a house for \$6000, agreeing to pay \$1000 down and \$1000 with interest at 4% on the unpaid balance at the end of each year for 5 yr. Money being worth 5%, compounded semiannually, what sum of money will liquidate the debt at once?

Exercise 48. Review of Chapters I-V

1. Expand $(1+i)^6$ and use the result as a formula to find the value of the accumulation factor 1.04^6, correct to four decimal places.

2. Find the value of 1.04^6 by the Binomial Theorem.

3. Simplify $(1+i)^{\frac{3}{2}} \times (1+i)^2 \times (1+i)^{\frac{1}{2}} \times 800$, and find the value of the result when $i = 0.04$.

4. Prove that the logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power.

5. Make a drawing of a slide rule set to check the items of a pay roll for which the rate is 35¢ an hour, indicating also the reading of wages for 42 hr.

6. By the aid of a slide rule determine whether the following statement is correct: $\frac{\$450 \times 0.04 \times 27}{365} = \$1.35$.

7. Find the limit of the sum of the infinite series v^{10}, v^{20}, v^{30}, $\cdots$, in which $v^n = 1/(1+i)^n$.

8. If the value of v^{10} in the series of Ex. 7 is 0.545, find the value of the limit of the sum of that series.

9. It is estimated that the life of a certain machine used by a manufacturing company is 10 yr., the scrap value being negligible. A new machine costs \$800. What sum put at compound interest at the present time at 4% will provide for the indefinite renewal of this asset?

We evidently have to find the present value of \$800 due in 10 yr., \$800 due in 20 yr., and so on indefinitely.

10. What is the difference between the amount of \$5000 due in 5 yr. at 5% at simple interest and the amount at compound interest, accumulated annually?

CHAPTER VI

EQUATION OF PAYMENTS

111. Commuting Obligations. It is occasionally desirable to exchange one or more obligations of unequal amounts due at future times for a series of equal installments due at equal intervals. Since money can be invested at the current rate of interest, an obligation of \$800 due in 2 yr. and one of \$1200 due in 4 yr. could not equitably be exchanged for two installments of \$1000 each, payable in 2 yr. and 4 yr. respectively; for the creditor would then have on hand at the end of the 4 yr. the interest accumulations of \$200 for 2 yr. in excess of the amount he would have by the first arrangement.

An equitable arrangement in the above case would be one by which, at the end of 2 yr. and at the end of 4 yr., there would be paid such installments that the creditor would have as much on hand at the end of 4 yr. as he would have had under the original conditions. Obviously, therefore, the installments being equal, the amount paid must depend on the rate of interest which the creditor can realize.

If money is worth 4%, the creditor in the case mentioned above could afford to accept the payment of \$996.04 at the end of 2 yr. and an equal payment at the end of 4 yr. As will be explained later in this chapter, by the original terms he would have, at the end of 4 yr., $1.02^2 \times \$800 + \$1200$, or \$2032.32. But $1.02^2 \times \$996.04 + \$996.04$ also amounts to \$2032.32.

The treatment of this subject has not been completely standardized in the commercial world. As given in this chapter it represents quite as much an ideal of proper methods as it does current practice.

112. Illustrative Example. A man owes \$800, due in 3 yr., and \$1200, due in 5 yr. He desires to exchange these obligations for a contract calling for two equal payments, one due in 2 yr. and the other in 4 yr. If money is worth 6% per annum, what sum should be paid as installments?

There are two general methods of attacking such a problem, the first being by a comparison of present values. In order that neither the debtor nor the creditor may have any advantage over the other it is necessary that *the present values of the exchanged obligations shall be equal.*

Let x = the number of dollars in the installment.

Then xv^2 = the present value of installment 1,

and xv^4 = the present value of installment 2.

But $800\,v^3$ = the present value of obligation 1,

and $1200\,v^5$ = the present value of obligation 2.

Using the table of values of v^n, pages 214–217, and remembering that the exchanged present values must be equal, we have

$$0.8899964\,x + 0.7920937\,x = \$671.70 + \$896.71.$$

Whence $x = \$932.42.$

As already stated, while a symbol like \$ is not necessary or even common in an equation, it may occasionally be used to advantage.

The second general method depends on the principle that the creditor must be able to accumulate as much at compound interest by the new method, when the last payment is made, as by the old one; that is, *the accumulated values must be equal.*

In the above problem, by this second plan the first installment can be invested for 3 yr. and the second for 1 yr.

By the terms of the indebtedness \$800 can be invested for 2 yr., while \$1200 earns no interest.

Hence $1.06^3\,x + 1.06\,x = 1.06^2 \times \$800 + \$1200,$

or $1.191016\,x + 1.06\,x = \$898.89 + \$1200.$

Solving, $x = \$932.42.$

113. Equation of Value. If a series of sums $X_1, X_2, X_3, \cdots$ due in $n_1, n_2, n_3, \cdots$ years respectively are severally equivalent to the sums $Z_1, Z_2, Z_3, \cdots$ due in $m_1, m_2, m_3, \cdots$ years respectively, we may discount separately all the sums and from their present values obtain the *equation of value.*

$$X_1v^{n_1} + X_2v^{n_2} + X_3v^{n_3} + \cdots = Z_1v^{m_1} + Z_2v^{m_2} + Z_3v^{m_3} + \cdots.$$

In all such cases compound interest is to be used. By replacing v^n by $1/(1+i)^n$ and multiplying both members of the equation by the highest power of $1+i$ found in the denominators of the various fractions, we may derive another form of the equation of value.

Exercise 49. Commuting Obligations

1. Money being worth 4%, commute a debt of $500 due in 3 yr. and one of $1500 due in 8 yr. into two equal installments due in 4 yr. and 6 yr. respectively.

2. Money being worth 5%, commute a debt of $1000 due in 3 yr., one of $1200 due in 5 yr., and one of $1500 due in 6 yr. into six equal annual installments due in 1 yr., 2 yr., $\cdots$, 6 yr.

3. A man desires to pay a debt of $1500 due in 3 yr. by making three equal annual payments, one due each year. Money being worth 5%, find the amount of each installment.

4. In Ex. 3 suppose that the man wished each successive installment to be $100 more than the preceding one; find the amount of each installment.

5. Money being worth 4%, find the equal payments that can be made in 3 yr., 4 yr., 5 yr., and 6 yr. respectively to equitably replace a single payment of $20,000 due in 6 yr.

6. Solve Ex. 5, using 5% as the current rate.

The student should notice the effect of an increase in the interest rate on the size of the installments.

7. Solve Ex. 5, using 6% as the current rate.

114. Equated Time for Small Debts. If a debtor owes debts due at different times and wishes to cancel them by a single payment without paying any interest, the date of this payment should be such that the interest lost by the creditor on the debts first due shall just balance the gain by receiving the other amounts before they are due. The time that should elapse to this date of payment is called the *equated time.*

After finding the equated time we may, if we wish, discount the payment for this length of time and pay at once. Transactions involving equated time are quite common in certain lines of foreign trade.

If $d_1, d_2, d_3, \cdots$ are debts due in $n_1, n_2, n_3, \cdots$ years respectively, and n is the number of years in the equated time, we have

Present value of debts $= d_1v^{n_1} + d_2v^{n_2} + d_3v^{n_3} + \cdots$;

Present value of payment $= (d_1 + d_2 + d_3 + \cdots)v^n$.

$$\therefore (d_1 + d_2 + d_3 + \cdots)v^n = d_1v^{n_1} + d_2v^{n_2} + d_3v^{n_3} + \cdots.$$

$$\therefore v^n = \frac{d_1v^{n_1} + d_2v^{n_2} + d_3v^{n_3} + \cdots}{d_1 + d_2 + d_3 + \cdots}.$$

Calling V the sum of the present values and D the sum of the debts, we have

$$v^n = (1+i)^{-n} = V/D;$$

whence
$$n = \frac{\log V - \log D}{\log v} = \frac{\log D - \log V}{\log(1+i)}.$$

For example, if a man owes \$2000 due in 3 yr., \$1200 due in 4 yr., and \$800 due in 6 yr., and money is worth 4%, we have

$$V = \$2000\,v^3 + \$1200\,v^4 + \$800\,v^6$$
$$= \$1777.99 + \$1025.77 + \$632.25 = \$3436.01,$$

and $D = \$4000.$

Hence
$$n = \frac{\log 4000 - \log 3436.01}{\log 1.04} = \frac{3.602060 - 3.536053}{0.017033} = 3.875.$$

Hence the equated time is 3.875 yr., or 3 yr. 10 mo. 15 da.

Exercise 50. Equated Time

1. Money being worth 6%, find the equated time for paying in one sum these debts: \$500 due in 2 yr., \$1000 due in 3 yr., \$1500 due in 4 yr., and \$800 due in 6 yr.

2. A man installed an engine, paying \$500 cash and agreeing to pay \$500 at the end of each year for 5 yr. Because of an unexpected increase in his net income he is able to make a single payment to cancel the obligation. How long after the date of purchase should he count the balance of \$2500 as due, money being worth 5%?

3. In Ex. 2, if the man desires to make a single payment at the end of 1 yr. from the date of purchase, how much must he pay?

Use logarithms to find the present value of the \$2500 balance due in the equated number of years from the date of purchase as found in Ex. 2.

4. Find the equated time for paying in one sum these debts: \$1000 due in 1 yr., \$2000 due in 3 yr., \$2500 due in 4 yr., and \$3000 due in 5 yr., money being worth 6%.

5. Find the equated time for paying four equal annual installments of \$300 each, due in 1 yr., 2 yr., 3 yr., and 4 yr. respectively, so as to discharge these obligations in one payment of \$1200, money being worth 5%.

6. Find the equated time in Ex. 5, money being worth 6%.

7. Money being worth 4%, find the equated time for paying in one sum the following debts: \$3000 due in 2 yr., \$4000 due in 3 yr., \$2500 due in 4 yr., and \$2500 due in 5 yr.

8. What sum of money invested now at 4% would amount to the sum of the debts at the equated time found in Ex. 7?

9. If money is worth 5%, find the equated time for paying in one sum these debts: \$8000 due to-day, \$3000 due in 2 yr., \$5000 due in 3 yr., and \$6000 due in 4 yr.

115. Equation of Value on a Simple-Interest Basis. While the method of § 113 represents scientific accuracy and is used by actuaries and investment calculators, especially when the times exceed an accumulation period for interest, business men commonly use a simple-interest basis.

In this case, the equating of present values gives a different result from that obtained by equating final values, as will be seen in solving the following problem:

Required to commute a debt of \$800 due in 3 yr. and one of \$1200 due in 5 yr. into two equal installments, one payable in 2 yr. and the other in 4 yr., money being worth 6% simple interest.

Equating present values, we have

$$\frac{x}{1.12} + \frac{x}{1.24} = \frac{\$800}{1.18} + \frac{\$1200}{1.30};$$

whence $$x = \$942.17.$$

Equating future values, that is, the amounts the creditor could accumulate at the end of the 5-year period, we have

$$1.18\,x + 1.06\,x = 1.12 \times \$800 + \$1200;$$

whence $$x = \$935.71.$$

If we used compound interest we should not have these differences. From the equation

$$\frac{x}{1.06^2} + \frac{x}{1.06^4} = \frac{\$800}{1.06^3} + \frac{\$1200}{1.06^5}$$

we have, using the table of values of v^n, pages 214–217,

$$0.8899964\,x + 0.7920937\,x = \$671.69544 + \$896.70984.$$

Adding, $$1.6820901\,x = \$1568.40528.$$

Hence $$x = \$932.41.$$

Since money really earns compound interest when the interest is promptly reinvested, the logical and scientific method of procedure demands the use of compound interest in calculations of this kind.

The student may compare the result with that found on page 100.

116. Equated Time with Simple Interest. Simple interest is ordinarily used in finding equated time. Proceeding on this basis, let $d_1, d_2, d_3, \cdots$ be the debts due in $n_1, n_2, n_3, \cdots$ years respectively, and let n be the number of years in the equated time. If we accumulate the debts to time n,

d_1 draws interest for $n - n_1$ years,

d_2 draws interest for $n - n_2$ years, and so on.

Since, at simple interest, $a = p(1 + it)$, the amounts are $d_1[1 + (n - n_1)i]$, $d_2[1 + (n - n_2)i], \cdots$. Hence

$$d_1 + d_2 + d_3 + \cdots = d_1[1 + (n - n_1)i] + d_2[1 + (n - n_2)i] + \cdots.$$

Expanding, reducing, and dividing by i, we have

$$(d_1 + d_2 + d_3 + \cdots)n = d_1n_1 + d_2n_2 + d_3n_3 + \cdots;$$

whence
$$n = \frac{d_1n_1 + d_2n_2 + d_3n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots}.$$

To find the equated time at simple interest, multiply each debt by the number of years it has to run and divide the sum of the products by the sum of the debts.

Using simple interest, find the equated time for the cancellation by a single payment of the following debts: \$300 due in 2 mo., \$420 due in 3 mo., and \$250 due in 4 mo.

By the above formula we have

$$n = \frac{300 \times \frac{1}{6} + 420 \times \frac{1}{4} + 250 \times \frac{1}{3}}{300 + 420 + 250}$$

$$= \frac{50 + 105 + 83\frac{1}{3}}{300 + 420 + 250}$$

$$= 0.2457.$$

Hence the equated time is 0.2457 yr., or 2 mo. 28 da.

If the debt of \$970 is paid at the end of 2 mo. 28 da., the creditor will lose the interest on \$300 for 28 da. and will gain the interest on \$420 for 2 da. as well as that on \$250 for 1 mo. 2 da.

Exercise 51. Commuting Obligations at Simple Interest

1. Find the equated time for paying in one sum a debt of \$500 due in 2 yr. and one of \$300 due in 5 yr.

2. Find the equated time for paying in one sum the following debts: \$250 due in $1\frac{1}{2}$ yr., \$200 due in 2 yr., and \$300 due in 3 yr.

3. Find the equated time for paying in one sum five successive equal annual installments of \$100, the first due in 1 yr.

4. Commute two debts, one of \$500 due in 3 yr. and one of \$800 due in 6 yr., into three equal installments due in 2 yr., 4 yr., and 6 yr. respectively, money being worth 6%.

5. Solve Ex. 4, money being worth 5% compound interest.

6. Find the equated time, on a simple-interest basis, for paying in a single amount a debt of \$800 due in 30 da., one of \$500 due in 60 da., and one of \$400 due in 90 da.

7. Check the correctness of Ex. 6 by comparing the totals of the simple interest gained or lost on the separate debts.

8. Find the equated time for settling in one payment of \$300 the following debts: \$100 due Mar. 1, \$100 due Mar. 15, and \$100 due Apr. 10.

9. In Ex. 8 calculate the interest gained and lost by the creditor, the settlement being made on the date found in the solution and the rate of interest being 4%.

10. If the debtor in Ex. 8 wishes to pay the full obligation of \$300 on Mar. 1 and is allowed a discount of 2% a month for payment before the equated time, how much must he pay in order to cancel the entire debt?

11. Find the equated time for paying in one sum 40 successive monthly installments of \$10 each, the first installment being due in 1 mo.

117. Equation of Accounts. If an account containing debit and credit items is to be settled, the balance should be reckoned as due at such a date as shall make the interest on the debit items to or from that date balance the interest on the credit items. Since the times involved are usually short, simple interest is used in the computation.

For example, if Y owes X \$50, due Jan. 11, and X owes Y \$25, due Jan. 21, the balance is \$25. If they settle on Jan. 1, a date which we shall show later how to find, Y has the use of \$25 for 20 da. before it is due and X has the use of \$50 for 10 da. before it is due. Whatever the interest rate may be, the amounts of the interest earned will balance, for $50 \times i \times 10 = 25 \times i \times 20$. Hence $50 \times 10 = 25 \times 20$, the two members being known as *dollar-day charges*.

In finding the equated time the sum of the dollar-day charges on the debit and credit sides of an account must be equal.

118. Illustrative Examples. 1. Find the equated time for the payment of the balance on the following account:

Dr.							*Cr.*
1920				1920			
Mar. 9	To mdse.	300		Mar. 20	By cash	247	25
May 12	"	473	60	May 11	By 30-day draft	400	
June 19	"	564	20	July 10	By cash	259	55

We select Mar. 9, from which to reckon, known as the *focal date*, and take each item to the nearest dollar.

Letting x be the number of days after Mar. 9, we have:

DUE	TIME	ITEM	PRODUCT	DUE	TIME	ITEM	PRODUCT
Mar. 9	x	300	$300x$	Mar. 20	$x-11$	247	$247(x-11)$
May 12	$x-64$	474	$474(x-64)$	May 11	$x-93$	400	$400(x-93)$
June 19	$x-102$	564	$564(x-102)$	July 10	$x-123$	260	$260(x-123)$

Equating sums of products, $1338x - 87{,}864 = 907x - 71{,}897$, whence $x = +37$, and the day of settlement is Mar. 9 + 37 da., or Apr. 15.

2. Find the equated time in the following account:

Dr.							*Cr.*
1922				1922			
Mar. 1	To mdse.	456	50	Mar. 15	By cash	250	
Apr. 20	"	479	85	May 9	By draft, 60 da.	379	75
June 10	"	430		May 31	By cash	560	20
July 5	"	575	15	July 11	"	545	

Letting x be the number of days after Mar. 1 for the settlement, we have

$$\begin{array}{ll} 457\,x & 250\,(x-14) \\ 480\,(x-50) & 380\,(x-129) \\ 430\,(x-101) & 560\,(x-91) \\ 575\,(x-126) & 545\,(x-132) \\ \hline \end{array}$$

Equating, $$1942\,x - 139{,}880 = 1735\,x - 175{,}420.$$

Therefore $$x = -172.$$

Hence we have, as the date of settlement, Sept. 10, 1921, or 172 da. before Mar. 1, 1922.

Exercise 52. Equation of Accounts

1. Find the equated time in the following account:

Dr.							*Cr.*
1923				1923			
Feb. 1	To mdse.	900	70	Feb. 5	By cash	700	70
May 10	"	850	70	May 5	"	200	

2. Find the equated time in the following account:

Dr.							*Cr.*
1921				1921			
Jan. 5	To mdse.	1200		Jan. 6	By cash	820	
Feb. 6	"	900		Feb. 11	"	700	
Mar. 10	"	850		Mar. 10	"	100	

Exercise 53. Review of Chapter VI

1. On purchasing a house a man borrowed $5000 from the Thrift Society and mortgaged the property to cover the loan. By the terms of his contract he was to pay $500 each year on the interest day together with interest at 5% on this sum from the date of the contract. If he should prefer to commute these unequal payments into ten equal annual payments without interest, what would be the amount of each installment?

2. A wholesale house accepted three notes in payment of a consignment of groceries, one for $100 maturing in 30 da., one for $150 maturing in 60 da., and one for $150 maturing in 90 da. If it is convenient to both parties to cancel the three notes at one time, allowing simple interest in finding the equated time, when could this be done without loss to either?

3. If A owes B the sum of $400 due Mar. 1, and B owes A the sum of $250 due Mar. 20, what is the equated time for settling the account?

4. If the account in Ex. 3 is settled Apr. 1 and money is worth 6%, what amount of interest should A pay in addition to the balance of $150?

5. A man bought $200 worth of feed from the Eureka Feed Company Jan. 1, and $250 worth Jan. 20. Jan. 5 he gave the company a note for $200 at 6% maturing in 30 da. Find the equated time for settlement.

6. If the balance in Ex. 5 is paid Mar. 1 and money is worth 6%, find the interest due on the balance.

7. Jan. 1 a merchant bought from a wholesale dealer a consignment of goods amounting to $450, terms 60 da. net. Mar. 1 he bought another consignment amounting to $350, terms 30 da. net. Feb. 1 he gave the dealer his note for $400 due in 30 da. Find the equated time for settling the balance.

Exercise 54. Review of Chapters I-VI

1. The horse power (H. P.) needed to propel a vessel through the water varies as the cube of the speed. If a speed of 12.4 knots is attained by a vessel with an indicated H. P. of 540, find by logarithms the H. P. needed to develop in the same vessel a speed of 18.6 knots.

2. Express in integral form the equation of the present value of 1, $v = 1/(1+i)^m$.

3. Express in integral form the equation

$$A = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right].$$

4. By the aid of logarithms find the value of x in the equation

$$500 \times 30 = \frac{1450 \times 144 \times \$1.37}{x}.$$

5. Check the result in Ex. 4 by using the slide rule.

6. By the Binomial Theorem find the value of 1.025^4 correct to four decimal places.

7. Using the result of Ex. 6, find the compound amount of $100 for 4 yr at $2\frac{1}{2}\%$, and find the compound interest.

8. Find the cash value of a purchase of a house and lot made on the following terms: $1000 down and $4000 to be paid in four installments of $1000 each, these installments being due at the end of each year for 4 yr. without interest, money being worth 6%.

9. In how many years will $5000 amount to $6500 at 5% compound interest?

10. A man wishes to commute three installments of $500 each, due in 2 yr., 4 yr., and 6 yr. respectively, into two equal installments due in 3 yr. and 6 yr. respectively. Money being worth 6% compound interest, find each installment.

CHAPTER VII

ANNUITIES

119. Annuity. A series of equal payments payable at regular intervals of time is called an *annuity.*

120. Kinds of Annuities. An annuity in which the first payment is payable at the end of the first interval is called an *ordinary annuity.*

An annuity in which the first payment is payable at the beginning of the first interval is called an *annuity due.*

The annual premiums paid in advance to an insurance company illustrate an annuity due.

An annuity beginning and ending on fixed dates, like interest payments on a bond, is called an *annuity certain.*

An annuity in which the payments are to begin or end at a time determined by some event whose occurrence cannot be exactly foretold is called a *contingent annuity.*

For example, the premiums on an ordinary life-insurance policy.

An annuity in which the first payment is payable after a definite number of payment intervals is called a *deferred annuity.*

An ordinary annuity deferred m intervals has the first payment made at the end of $m+1$ intervals.

An annuity in which the payments are to be continued forever is called a *perpetuity.*

An annuity in which the payments are allowed to accumulate to the end of the series is called a *foreborne annuity.*

In this case the sum is called the *amount* or *final value* of the annuity.

121. Present Value. The sum of the present values of all the payments of an annuity is called the *present value* of the annuity. This may be expressed as the sum which, put at interest at the current effective rate, will amount to the total annuity when the last payment is due.

For example, if money is worth 6%, the present value of an annuity of $800 a year for 3 yr. is $\frac{\$800}{1.06}+\frac{\$800}{1.06^2}+\frac{\$800}{1.06^3}$, or $2138.42.

122. Annual Rent. The sum of the payments on an annuity payable in a year is called the *annual rent* of an annuity.

As the amount paid per year is usually designated, we speak of "an annuity of $800 per year." In the illustration of § 121, $800 is the annual rent of the ordinary annuity sold for $2138.42, its present value.

123. Notation. In general in English-speaking countries the italic letters *a*, *s*, are used for ordinary annuities, and heavy roman letters **a**, **s**, for annuities due, thus:

$s_{\overline{n}|}$ represents the amount or final value of an ordinary annuity of 1 per annum, payments to continue for n years.

The symbol $\overline{}|$ indicates that the period is definite or certain.

$a_{\overline{n}|}$ represents the present value of an ordinary annuity of 1 per annum for n years.

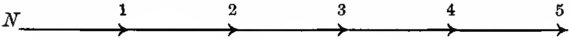
It is evident that if the annual rent R of §§ 121, 122 had been $1, instead of $800, the result in each case would have been $\frac{1}{800}$ as great, or $2.6730119.

$\mathbf{s}_{\overline{n}|}$ represents the final value of an annuity due of 1 per annum for n years.

$\mathbf{a}_{\overline{n}|}$ represents the present value of an annuity due of 1 per annum for n years.

$m|a_{\overline{n}|}$ and $m|s_{\overline{n}|}$ represent the present value and the final value respectively of annuities of 1 per annum for n years, each deferred m years. The first payment is thus due $m+1$ years from date.

124. Final Value of an Ordinary Annuity. Let us consider an annuity of \$300 per annum, payable annually for 5 yr.



Representing time by a straight line, the diagram shows the operation of an ordinary annuity certain of five payments. If the annuity is acquired now, the date being N in the figure, the first payment will be due at 1, the second at 2, and so on, each interval representing a year.

It is evident that the payment due at 1 will accumulate interest for 4 yr. if left unpaid until 5. Its amount will be $1.06^4 \times \$300$ while the amount of that due at 2 will be $1.06^3 \times \$300$, and so on.

The sum, A_5, of these payments is found by adding, thus:

$$A_5 = (1 + 1.06 + 1.06^2 + 1.06^3 + 1.06^4) \times \$300.$$

It is evident that the quantity in parentheses is the amount of an annuity of 1 per annum for 5 yr. at 6%, or that $A_5 = Rs_{\overline{5}|}$. In general, $A_n = Rs_{\overline{n}|}$. Hence we need to find an expression for $s_{\overline{n}|}$ at the rate i.

125. Final Value of 1 per Annum. Let an ordinary annuity of 1 per annum be payable for n years, and let the rate of interest be i per annum. As in § 124, we have

$$\text{Amount of first payment} = (1+i)^{n-1}.$$

$$\text{Amount of second payment} = (1+i)^{n-2}.$$

. . . .

$$\text{Amount of } n\text{th payment} = 1.$$

Adding, $s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}.$

In this geometric series, $a = 1$, $r = 1 + i$, $n = n$.

Since the sum of a geometric series is $\dfrac{ar^n - a}{r-1}$, we have

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i} = \frac{1}{i}[(1+i)^n - 1].$$

A table of values of $s_{\overline{n}|}$ will be found on page 218.

126. Final Value of R per Annum. It is evident that if the annual rent is R per annum, instead of 1, the amount in n years is $R(1+i)^{n-1}+R(1+i)^{n-2}+\cdots+R(1+i)+R$, or $R[(1+i)^{n-1}+(1+i)^{n-2}+\cdots+(1+i)+1]$, which is equal to $RS_{\overline{n}|}$. If we designate the amount of R per annum for n years by A_n, we have

$$A_n = Rs_{\overline{n}|} = \frac{R}{i}[(1+i)^n - 1].$$

127. Illustrative Examples. 1. A trustee deposited \$200 in a savings bank at the end of each year for 7 yr., the bank paying 4% per annum, accumulated annually. Find the amount to his credit after he makes the seventh deposit.

Since $R = \$200$, $i = 0.04$, $n = 7$, we have $A_7 = \frac{\$200}{0.04}(1.04^7 - 1)$, or $\$200 \times s_{\overline{7}|}$ at 4%.

Using the table, page 220, $A_7 = 7.8982945 \times \$200 = \$1579.66$.

If a table of final values of 1 were not available, we could evaluate the first form and obtain

$$A_7 = \frac{\$200}{0.04}(1.04^7 - 1) = (1.3159318 - 1) \times \$5000 = \$1579.66, \text{ as before.}$$

2. If the deposits in Ex. 1 drew interest at 4% nominal, payable semiannually, what would be the final value?

The series becomes $1.02^{14} \times 200 + 1.02^{12} \times 200 + \cdots + 1.02^2 \times 200 + 1$.

In general, $(1+i)^n$ is replaced by $\left(1+\frac{N}{m}\right)^m$, where N is the nominal rate, and m the number of times interest is compounded per annum; and i is replaced by $\left(1+\frac{N}{m}\right)^m - 1$. Our formula for $s_{\overline{n}|}$ becomes

$$s_{\overline{n}|} = \frac{\left(1+\frac{N}{m}\right)^{mn} - 1}{\left(1+\frac{N}{m}\right)^m - 1}.$$

Here $\quad A_7 = \frac{1.02^{14} - 1}{1.02^2 - 1} \times \$200 = \frac{1.3194788 - 1}{1.0404 - 1} \times \$200$

$$= \$63.89576 \div 0.0404 = \$1581.58.$$

128. Annual Rent; the Interest Converted p Times a Year. The case in which the rent is paid p times a year, the interest being converted p times and the intervals coinciding, is of especial importance in commercial calculations. It is commonly assumed that interest may be converted when annuity payments are received, as in the case of bond valuations to be explained later. Our annuity of R per annum, payable p times a year, interest at the rate of i per annum, converted p times a year, thus amounts to an annuity of pn intervals, R/p being the rent per interval and the rate per interval being i/p.

Thus an annuity of $300 a year for 5 yr., payable semiannually, the interest being 4% a year, payable semiannually, will have the same final value as an annuity of $150 a year for 10 yr. at 2%.

A bond for $1000 bears interest at 6%, payable semiannually on Jan. 1 and July 1, and when the interest payments are received they are deposited in a savings bank which pays 4%, compounded semiannually. Find the amount that will be to the credit of the depositor at the end of 8 yr., after the last interest payment has been deposited by using the tables for $s_{\overline{n|}}$ and also the expanded form of $s_{\overline{n|}}$.

There will be 16 interest payments of $30 each, the payments corresponding to the annual rent of an annuity of 16 terms, bearing interest at 2%. Hence

$$A_{16} = \$30 \times s_{\overline{16|}} \text{ at } 2\% = 18.6392853 \times \$30 = \$559.18.$$

If we use the expanded form of $s_{\overline{16|}}$ and the compound interest table, we have

$$A_{16} = \frac{30}{0.02}(1.02^{16} - 1) = 1500 \times 0.3727857 = 559.18.$$

Since the interest on many classes of bonds is payable Jan. 1 and July 1, it is possible to reinvest in similar securities the interest thus received on these dates. The interest is thus compounded semiannually and is converted at the same intervals as before.

Exercise 55. Final Value of an Annuity

1. Find the amount of an annuity of \$800 a year for 19 yr., money being worth 4%, compound interest.

2. Find the amount of \$800 a year for 19 yr., payable semiannually, money 4% nominal, convertible semiannually.

3. The rent of a building is \$2000 a year, payable June 30 and Dec. 31, and the money is deposited by a guardian with a trust company that pays 4% nominal, compounded semiannually. This is continued 11 yr., until the ward reaches his majority. What sum will then be on deposit?

4. Find the amount of a forborne annuity of \$1200 a year, payable annually, for 21 yr., if money is worth 4% per annum.

5. A man having won a suit for a pension which has been denied him 3 yr., the third payment being now due, compound interest at 4% per annum is allowed him by the court. He is offered \$3700 for the three payments of \$1200 each. Will it be financially to his advantage to accept the offer?

6. Find the final value of an annuity of \$1200 a year, left unpaid for 11 yr., money being worth 5%, compound interest.

7. At the end of each year for 10 yr. a man deposited \$500 in a savings bank that pays 4% per annum, payable semiannually on Jan. 1 and July 1. Find the amount to his credit.

8. In Ex. 7, if the man deposited \$250 each half year during 10 yr., how much more would he have to his credit than if he deposited \$500 annually?

9. Using the formula for S in a geometric series, find the sum of $1.04^6 + 1.04^5 + 1.04^4 + \cdots + 1.04$.

10. Given $(1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)$, this being the final value $s_{\overline{n|}}$ of an annuity due of 1 per annum, find the sum.

129. Final Value of an Annuity Due. Since the annual rent of an annuity due is paid at the beginning instead of the end of each year, the interest accumulation is greater than in the case of an ordinary annuity. This will be seen, for example, by comparing the amount of an ordinary annuity of 1 per annum with the amount of an annuity due of 1 per annum for 4 yr. at 5% compound interest.

For the ordinary annuity, $s_{\overline{4}|} = 1.05^3 + 1.05^2 + 1.05 + 1.$

For the annuity due, $\mathbf{s}_{\overline{4}|} = 1.05^4 + 1.05^3 + 1.05^2 + 1.05.$

That is, $\mathbf{s}_{\overline{4}|} = 1.05 \times s_{\overline{4}|}.$

Hence $\quad s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1,$

and $\quad \mathbf{s}_{\overline{n}|} = (1+i)^{n} + (1+i)^{n-1} + \cdots + (1+i)^2 + (1+i),$

and so $\quad \mathbf{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}.$

Evidently, too, the formula for $\mathbf{s}_{\overline{n}|}$ may be written

$$\mathbf{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1.$$

The last of these formulas is useful in connection with the tables. Thus, to find $\mathbf{s}_{\overline{6}|}$ at 4% we look in the table of $s_{\overline{n}|}$ at 4% for $n = 7$, finding 7.8982945. Subtracting 1, we have $\mathbf{s}_{\overline{6}|} = 6.8982945$.

For the annual rent R, the final value A_n of an annuity due is given by the formula $A_n = R\mathbf{s}_{\overline{n}|}$.

Exercise 56. Final Value of an Annuity Due

1. A life-insurance company receives an annual premium of \$26.23 for 20 yr. for a 20-payment life policy. If the company accumulates funds at 4% compound interest, how much will it accumulate at the end of 20 yr.? at the end of 35 yr.?

2. What is the final value of an annuity due of \$100 a year for 30 yr. at 4% compound interest?

3. Find the amount of an annuity due of \$500 a year for 25 yr. at 4% per annum, payable semiannually?

130. Present Value of an Ordinary Annuity. If a payment of R is due at the end of 1 yr., money being worth i, the present value is Rv, where $v = 1/(1+i)$; R due in 2 yr. is worth Rv^2; and so on. Denoting the present value by A_0,

$$A_0 = Rv + Rv^2 + Rv^3 + \cdots + Rv^n$$
$$= R(v + v^2 + v^3 + \cdots + v^n).$$

The series in parentheses is evidently $a_{\overline{n}|}$, the present value of an annuity of 1 per annum payable for n years.

Hence $$\boldsymbol{A_0 = Ra_{\overline{n}|}}.$$

131. Value of $a_{\overline{n}|}$, Payments Annually, Interest Annually. By definition, $$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n.$$

The right member of this equation is a geometric series.

Hence $$a_{\overline{n}|} = \frac{v^{n+1} - v}{v - 1} = \frac{v - v^{n+1}}{1 - v}.$$

Dividing numerator and denominator by v, and substituting,

$$a_{\overline{n}|} = \frac{1 - v^n}{\frac{1}{v} - 1} = \frac{1 - v^n}{i} = \frac{1}{i}\left[1 - \frac{1}{(1+i)^n}\right].$$

Our formula for A_0 now becomes

$$\boldsymbol{A_0 = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right]}.$$

A table of values of $a_{\overline{n}|}$ is given on page 222.

Again, we see that by the second definition of the present value of an annuity A_0 is such a sum as, put at compound interest for the time and rate, will amount to the final value.

Hence $$A_0(1+i)^n = \frac{R}{i}[(1+i)^n - 1].$$

Dividing, $$A_0 = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right].$$

132. Illustrative Examples. 1. If I sell the right to receive \$300 rent at the end of each year, for 5 yr., how much should I receive if money is worth 5%, compound interest?

We have $A_0 = \$300 \times a_{\overline{5}|}$ at $5\% = 4.3294767 \times \$300 = \$1298.84$.

2. If money is worth 4% per annum, compounded semiannually, what is the present value of an annuity of \$300 to be paid at the end of each year for 5 yr.?

If the interest were paid annually, we should have by § 131

$$\frac{R}{i}\left[1-\frac{1}{(1+i)^n}\right] = \frac{\$300}{0.04}\left[1-\frac{1}{1.04^5}\right].$$

Since the interest is paid semiannually, the value of $1 + i$, instead of being 1.04, becomes 1.02^2 or 1.0404, and n becomes 10. Therefore we have

$$A_0 = \frac{\$300}{0.0404}\left(1-\frac{1}{1.02^{10}}\right) = \frac{\$300}{0.0404}(1-0.8203483)$$

$$= \frac{\$300}{0.0404} \times 0.1796517 = \$1334.05.$$

3. Find the present value of an annuity of \$200 a year for 6 yr., payable semiannually, the rate of interest being 4% per annum, convertible semiannually.

Since there are 12 payments of \$100 each and money is worth 2% per period, the present value is the same as that of an annuity of \$100 a year, payable annually for 12 yr., money being worth 2% per annum. Hence

$$A_0 = \frac{R}{2} a_{\overline{12}|} \text{ at } 2\% = 10.5753412 \times \$100 = \$1057.53.$$

In general, if an annuity of R per annum is payable m times a year, and the interest rate is N per annum, interest being payable m times a year, we have

$$A_0 = \frac{R}{m} \times a_{\overline{mn}|} \text{ at rate } \frac{N}{m}.$$

In such business calculations as bond valuation, when payments are made semiannually, it is assumed that interest is convertible semiannually, the payment and conversion intervals coinciding.

Exercise 57. Present Value of an Annuity

1. What is the present value of an annuity of $800 a year for 15 yr., money being worth 4%, payable annually?

2. Find the present or cash value of a series of eight annual installments of $400 each, money being worth 6% effective.

3. Find the present value of an annuity of $400 a year, payable annually for 50 yr., money being worth 6% effective.

4. Find the cash value of an annuity of $400 a year, payable semiannually for 25 yr., money being worth 4% nominal, convertible semiannually.

5. Find the cash value of an article that can be bought for $10 down and $5 a month for 16 mo., money being worth 4% a month to the selling house, or 48% nominal, convertible monthly.

6. Find the present value of $400 a year, payable at the end of each year for 5 yr., money being worth 4% nominal, payable semiannually.

7. Find the present value of the payments in Ex. 6, money being worth 4% effective.

8. A man contracts to buy a farm for $1000 down and $4800 in twelve equal annual installments. In place of the terms of the contract, the seller offers to accept $4800 in cash for the farm. Money being worth 6% effective, which is the better plan for the purchaser, and how much better?

9. Find the sum of this series, as indicated:

$$1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots+\frac{1}{(1+i)^{n-1}}.$$

10. Find the face of a debt due to-day that can be satisfied by assigning to the creditor an annual income of $800, payable annually, money being worth 5%.

133. Present Value of an Annuity Due. The relation of the present value of an annuity due of 1 per annum to an ordinary annuity for the same number of years, the interest rate being the same, can be understood from the following:

1. Compare the present value of an ordinary annuity with that of an annuity due of 1 per annum for 4 yr., money being worth 6% compound interest.

We have
$$a_{\overline{4}|} = \frac{1}{1.06} + \frac{1}{1.06^2} + \frac{1}{1.06^3} + \frac{1}{1.06^4},$$

and
$$\mathbf{a}_{\overline{4}|} = 1 + \frac{1}{1.06} + \frac{1}{1.06^2} + \frac{1}{1.06^3}.$$

In general,
$$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^{n-1} + v^n,$$

and
$$\mathbf{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \cdots + v^{n-1}.$$

Hence we conclude that
$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|},$$

and
$$A_0 = R\,\mathbf{a}_{\overline{n}|} = R(1 + a_{\overline{n-1}|}).$$

2. Find the present value of an annuity due of $50 per year for 5 yr., money being worth 4% compound interest.

Here $A_0 = (1 + a_{\overline{4}|}$ at $4\%) \times \$50 = 4.6298952 \times \$50 = \$231.49$.

The factor 4.6298952 is found on page 224 from the table of values of $a_{\overline{n}|}$ for 4 yr. at 4%, 1 being added.

Exercise 58. Present Value of an Annuity Due

1. A man has the rent of a building amounting to $3000 a year, payable annually in advance. Money being worth 6% compound interest, and the year's rent being now due, find the present value of 11 payments.

2. Find the present value of an annuity due of $49.63 a year for 20 yr., money being worth 4% compound interest.

3. Find the cash value to an insurance company of 20 premiums of $23.89 a year, payable in advance, money being worth 4% compound interest.

134. Present Value of a Deferred Annuity. If an ordinary annuity of 1 per annum does not begin until after the lapse of m years and then continues n years, its present value, $m|a_{\overline{n}|}$, may be found in either of the two following ways:

1. After m years the annuity will be an ordinary annuity and its value will be $a_{\overline{n}|}$, and so its present value is $a_{\overline{n}|}v^m$.

2. From the two equations

$$a_{\overline{m+n}|} = v + v^2 + v^3 + \cdots + v^m + v^{m+1} + \cdots + v^{m+n}$$

and

$$a_{\overline{m}|} = v + v^2 + v^3 + \cdots + v^m$$

we see that $a_{\overline{m+n}|} - a_{\overline{m}|} = v^{m+1} + v^{m+2} + \cdots + v^{m+n}$.

The second member of the last of these equations is evidently the present value of an annuity of 1 for n years, the first payment being due in $m+1$ years.

Therefore $$m|\mathbf{a}_{\overline{n}|} = \mathbf{a}_{\overline{m+n}|} - \mathbf{a}_{\overline{m}|}.$$

Find the present value of the annual rental of $500 of a building for 5 yr., the first rental being due in 6 yr. and money being worth 4% compound interest.

We have $m = 5$ and $n = 5$, and therefore

$$m|a_{\overline{n}|} = 5|a_{\overline{5}|} = a_{\overline{10}|} - a_{\overline{5}|} \text{ at } 4\%.$$

From the table,

$$a_{\overline{10}|} - a_{\overline{5}|} = 8.1108958 - 4.4518223 = 3.6590735.$$

Hence the present value is 3.6590735 × $500, or $1829.54.

Exercise 59. Present Value of a Deferred Annuity

1. Find the present value of an annuity of $50 for 10 yr., deferred 4 yr., money being worth 4% compound interest.

2. Find the present value of an annuity of $500 for 18 yr., deferred 5 yr., money being worth 5% compound interest.

3. Find the present value of an annuity of $350 for 25 yr., deferred 5 yr., money being worth 5% compound interest.

135. To find the Term of an Annuity Certain. 1. A man borrows \$4500 which he agrees to repay in installments of \$800 with interest at 6% at the end of each year. How long will it take to pay the debt?

The installments constitute an annuity of annual rent \$800, rate 6%, whose present value must be equal to the face of the debt. Hence, § 131,

$$\$4500 = \frac{\$800}{0.06}\left(1 - \frac{1}{1.06^n}\right).$$

Simplifying, $$1.06^n = \frac{8}{5.3}.$$

Hence $$n = \frac{\log 8 - \log 5.3}{\log 1.06} = \frac{0.178814}{0.025306}.$$

This gives $n = 7.066$, the number of years.

2. How long will it take an investment of \$400 made at the end of each year at 4% to accumulate \$5000?

We have, § 126, $$\$5000 = (1.04^n - 1)\frac{\$400}{0.04}.$$

Simplifying, $$1.04^n = 1.5.$$

Hence $$n = \frac{\log 1.5}{\log 1.04} = \frac{0.176091}{0.017033}.$$

This gives $n = 10.338$, the number of years.

Exercise 60. Term of an Annuity Certain

1. If a city borrows \$100,000 at 4% per annum, and desires to repay principal and interest in annual installments of \$10,000 each, how many years will it take to do it?

2. If I invest \$250 at the end of each year at 6%, compound interest, how long will it take to accumulate \$5000?

3. A real estate company has houses for sale at \$6000 each, cash. If it desires to realize 6% compound interest on its money, how long will be needed to pay for a house by annual installments of \$600 each, payable at the beginning of each year?

136. Finding the Annual Rent. If an annuity is to run n years and its final value is to be A_n, from the formula $A_n = Rs_{\overline{n|}}$ we have the formula

$$R = A_n \cdot \frac{1}{s_{\overline{n|}}}. \tag{1}$$

This is known as the *sinking fund equation.*

If the present value A_0 is given, from the formula $A_0 = Ra_{\overline{n|}}$ we have the formula

$$R = A_0 \cdot \frac{1}{a_{\overline{n|}}}. \tag{2}$$

137. Illustrative Examples. 1. If a man borrows $500 to be repaid at 5% compound interest in four equal annual installments, what is the amount of each installment?

Since the installments cover principal and interest, their present value must be equal to the sum borrowed. Therefore

$$R = \$500 \cdot \frac{1}{a_{\overline{4|}}} \text{ at } 5\% = \$500 \times 0.2820118 = \$141.01.$$

2. If a $150,000 issue of bonds is to be redeemed in 20 yr., and money can be accumulated at 4% compound interest, what sum must be raised annually to meet the cost?

Since $A_n = \$150{,}000$, $i = 0.04$, and $n = 20$, we have

$$R = \$150{,}000 \cdot \frac{0.04}{1.04^{20} - 1} = \$150{,}000 \times 0.0335817 = \$5037.26,$$

the value 0.0335817 being found from the tables.

If the bonds bear interest at 5%, $7500 must be added to this sum annually, making the annual burden $12,537.26.

Exercise 61. Finding the Annual Rent

1. A man pays a debt of $6500 with interest at 5% in six equal annual installments. Find the amount of each.

2. How much must be deposited in a savings bank on Jan. 1, the bank paying 4% nominal, compounded semiannually, in order to accumulate $3000 at the end of 5 yr.?

Exercise 62. Review of Chapter VII

1. A 20-year endowment policy for $1000 for a man at the age of 35 in a certain insurance company costs $42.89, the premium being paid annually in advance. If the company accumulates its funds at 4% effective, how much will it accumulate from these payments by the end of the 20 yr.?

2. A corporation borrowed $100,000, which it agreed to repay, principal and interest at 5%, in ten equal annual installments. Find the annual rent of this annuity.

3. A certain railroad company borrowed $2,000,000 on bonds which mature in 20 yr. The bonds call for the establishment of a sinking fund, to which the railroad company is to make 20 equal annual payments. The payments are invested at 4%, compound interest. How large an annual contribution to the sinking-fund annuity must be made to satisfy the debt at the end of the 20 yr.?

4. If the bonds in Ex. 3 bear interest at 5%, how large a total contribution from the net income must be made by the company on account of the bond issue?

5. A man sets aside annually $2400 from the proceeds of his business, investing this at 4% compound interest. With the amount he thus accumulates in 10 yr. he intends to renew his equipment of machines. How much will he have in the fund at the end of the 10 yr.?

6. Find the present value of an annuity due of $25 a year for 10 yr., money being worth 4% compound interest.

7. Prove that $\dfrac{1}{a_{\overline{n|}}} - i = \dfrac{1}{s_{\overline{n|}}}$.

8. If money is worth 6%, compound interest, payable semiannually, how long will it take a series of semiannual installments of $200 each to amount to $2300?

Exercise 63. Review of Chapters I–VII

1. Using the formula $A = P(1+i)^n$, find by the aid of logarithms the compound amount of $5000 for $3\frac{1}{4}$ yr. at $2\frac{1}{4}\%$.

2. By the aid of the slide rule calculate the interest on $300 for 35 da. at $4\frac{1}{2}\%$, using 360 da. to the year.

3. Make a rough drawing of a slide rule set to change cubic inches to gallons or gallons to cubic inches. Indicate the scales on which numbers of gallons and numbers of cubic inches respectively are to be read.

4. Find the value of $\frac{1.06^0 \times 35}{1.06^4} \times \$550$.

5. Using the formula for the sum of an infinite series, find the present value of $800 due in 10 yr., $800 due in 20 yr., and so on forever, money being worth 4% compound interest.

6. Find the present value of 1 per annum, paid indefinitely, money being worth 4%, compound interest.

7. Find the present value of a perpetual series of payments of 1 per annum, the rate of compound interest being i.

8. If the present value of the perpetuity of 1 per annum found in Ex. 7 is designated by a_∞, the value of a perpetuity of R per annum would be Ra_∞. Write the equation for Ra_∞.

9. Using the results of Exs. 7 and 8, find the present value of a perpetual series of payments of $800 a year, money being worth 5% compound interest.

10. Find by logarithms the value of $\frac{1}{1.05^6}$ and compare the result with the value given in the table for v^n on page 216.

11. If A accepts the assignment of the rent of a building which amounts to $1200 a year in payment of a debt of $5000, how many years' rent should be assigned to him to pay the debt, money being worth 6% compound interest?

CHAPTER VIII

AMORTIZATION

138. Amortization and Sinking Fund. The process of paying off or liquidating any obligation is called *amortization.*

For example, if a man borrows $5000 to be repaid with interest at 5% in 10 equal annual installments, the paying of $647.52 annually will, as can be shown by the method given on page 128, amortize the debt.

The interest due at the end of 1 yr. is $250, and if $647.52 is paid, the principal is reduced $397.52, leaving $4602.48.

The interest at the end of 2 yr. is 5% of $4602.48, or $230.12, and if $647.52 is paid, the principal is reduced $417.40, leaving $4185.08, and so on. When the last installment of $647.52 is paid, it will be found that the debt is entirely satisfied.

When the face of a debt is not reduced by the annual payments, only the interest being paid as it becomes due, it is customary to accumulate a sum for the payment of the principal when it shall become due. This is done by setting aside annually such installments as will amount, with their accumulations at compound interest, to the face of the debt at the end of the stated time. Such installments constitute what is called a *sinking fund.*

That is, the amortization of a debt is commonly provided for by means of a sinking fund.

A schedule showing the annual payments and their appropriation to the payment of interest and the partial payment of principal is called an *amortization schedule*, and the same term is applied to a schedule showing the accumulation of a sinking fund.

139. Principal and Interest in Equal Installments. When the principal or face of the debt is gradually extinguished by periodic payments which not only cover the interest but also decrease the principal, it is plain that the creditor receives as an equivalent to the principal and interest a series of equal periodic payments, or an annuity. The one is evidently equivalent to the other at the time of the exchange. Hence

When principal and interest are paid in equal periodic payments, the present value of the annuity is the face of the debt.

Two cases now arise, and these are considered in §§ 140 and 144.

140. Determining the Periodic Payment. Since the face of the debt is equal to the present value of the annuity, and the time and rate are known, the annual rent, R, of the annuity is easily calculated from the equation $A = Ra_{\overline{n}|}$. For example: A man borrowed $6000, agreeing to repay principal and interest at 6% in eight equal annual installments. What is the annual payment?

The annual payment is evidently A_0, and

$$A_0 = R \cdot a_{\overline{8}|} \text{ at } 6\%.$$

Hence $$R = A_0 \cdot \frac{1}{a_{\overline{8}|}} \text{ at } 6\%.$$

Substituting, $$R = \$6000 \times 0.1610359 = \$966.215.$$

By leaving the annual payment in the form $966.215 we mean that the annual payments alternate $996.22 and $996.21.

As shown on page 129, at the end of the first year the man owes $360 interest, and when he pays $966.22, he pays the interest and $606.22 of the principal, leaving $5393.78 still due. This $5393.78 bears 6% interest, making the interest due at the end of the second year $323.63. When he pays $966.21 he pays the interest and $642.58 of the principal, leaving his indebtedness for the third year $4751.20. The interest on this indebtedness is $285.07, leaving a balance of $681.15 to reduce the face of the debt. This process is then continued during the remaining years.

141. Amortization Schedule. We shall now consider in detail an amortization schedule, taking the example given in § 140, the amortization of a debt of $6000 with interest at 6% in eight equal annual installments. The detailed amortization schedule is as follows:

Year	Principal, Beginning of Year		Interest for the Year		Annual Payments (Annuity)		Principal paid at End of Year	
1	6000		360		966	22	606	22
2	5393	78	323	63	966	21	642	58
3	4751	20	285	07	966	22	681	15
4	4070	05	244	20	966	21	722	01
5	3348	04	200	88	966	22	765	34
6	2582	70	154	96	966	21	811	25
7	1771	45	106	29	966	22	859	93
8	911	52	54	69	966	21	911	52
Totals	28828	74	1729	72	7729	72	6000	

As stated in § 140, the alternation of \$966.22 and \$966.21 in the annual payments is due to the fact that $R = \$966.215$.

If the work is correct, the sum of the items in the column of "Principal paid at end of year" will be equal to the total indebtedness, in this case \$6000.

Since the items in the interest column are found by multiplying the corresponding principals by the rate, we may check this column by using the algebraic principle that $rp_1 + rp_2 + rp_3 + \cdots = r(p_1 + p_2 + p_3 + \cdots)$. That is, 6% of \$28,828.74 should be equal to \$1729.72, as it is.

A further check is available in the fact that the sum of the payments on principal and the payments of interest is equal to the total annuity payments; that is, $\$6000 + \$1729.72 = \$7729.72$.

Another check, applicable as we proceed in the calculation, is available in the formula $A_0 = Ra_{\overline{n|}}$. For example, for the beginning of the fifth year we have $A_0 = Ra_{\overline{4|}} = \$966.215 \times 3.4651056 = \$3348.04$, which agrees with the outstanding principal.

142. Illustrative Examples. **1.** A man bought a piece of land, paid \$2000 down, and agreed to pay the balance of \$4000 with interest at 6% in four equal annual installments, the first installment to be paid 2 yr. from the date of purchase. Find the installment and construct the schedule.

At the end of 1 yr. he owes \$4240, and this obligation is to be met by an ordinary annuity of four terms; that is, by an annuity in which the first payment is due a year later. We therefore have $i = 6\%$, $n = 4$, and $A_0 = \$4240$.

Hence
$$R = \$4240 \cdot \frac{1}{a_{\overline{4}|}} \text{ at } 6\%$$
$$= \$4240 \times 0.2885915 = \$1223.628.$$

The detailed schedule is as follows:

Year	Principal, Beginning of Year		Interest for the Year		Annual Payments (Annuity)		Principal paid at End of Year	
1	4240		254	40	1223	63	969	23
2	3270	77	196	25	1223	63	1027	38
3	2243	39	134	60	1223	63	1089	03
4	1154	36	69	26	1223	62	1154	36
Totals	10908	52	654	51	4894	51	4240	

2. If a corporation wishes to borrow \$10,000 on a 5% basis, paying principal and interest in equal annual installments, how much more must be set aside annually if the amortization is accomplished in 8 yr. than if it runs 10 yr.?

In each case $A_0 = \$10{,}000$.

For the 8-year period, $R = \$10{,}000 \cdot \dfrac{1}{a_{\overline{8}|}}$ at 5%.

For the 10-year period, $R = \$10{,}000 \cdot \dfrac{1}{a_{\overline{10}|}}$ at 5%.

Hence the difference is $\$10{,}000 \cdot \left(\dfrac{1}{a_{\overline{8}|}} - \dfrac{1}{a_{\overline{10}|}}\right)$,

or \$10,000 (0.1547218 – 0.1295046), which is equal to \$252.17.

Exercise 64. Finding the Periodic Payment

1. A city borrows \$100,000 to be repaid, principal and interest at 4%, in 20 equal annual installments. What annual tax must be levied to amortize the debt during the 20 yr.?

2. If the assessed valuation of the city in Ex. 1 is \$18,300,000, find the increase in the tax rate necessary to repay the debt as agreed. How much will A's tax be increased if he pays on property assessed at \$8200?

3. A man buys a house for \$8000, paying \$2000 in cash and agreeing to pay the balance with interest at 6% in six equal annual installments. What is the annual payment?

4. Construct the amortization schedule of a debt of \$10,000 in five equal annual installments, including interest at 5%.

5. If a town borrows \$60,000 to be repaid with interest at 4% in 20 equal annual installments, what principal remains outstanding when the ninth installment is paid?

6. I buy a piano for \$500, paying \$50 down and agreeing to pay the balance with interest at 6% in 24 equal monthly installments. How much must I pay per month?

7. Construct the amortization schedule for the payment of a debt of \$3000 with interest at 6%, payable semiannually, in 12 equal semiannual installments.

8. How much larger will be the annual charge to amortize a debt of \$50,000, with interest at 4%, in 10 equal annual installments than in 15 equal annual installments?

9. The J. C. Ryan Co. negotiates a loan for \$12,500 at 5%, the principal and interest to be repaid in 3 yr. from the income. The company borrows the money on three notes for such amounts that the total annual charges for interest and for taking care of the notes are approximately equal. Find the face of each note.

143. Amortization of Principal and Interest. If a corporation, municipal or private, borrows money for a long period, it usually does so by issuing bonds. These bonds are usually issued in denominations of $1000, $500, or $100. The amortization schedule is ordinarily arranged so as to pay the interest on the outstanding bonds and to retire a certain number each year, the principal repaid being a multiple of $1000, $500, $100, or whatever the denominations may be.

Suppose that a school board wishes to borrow $60,000, issuing 60 bonds of $1000 each, and wishes to begin retiring these bonds at the end of 21 yr. and to complete the liquidation in 30 yr. If the tax levies for the 10 yr. of bond retirement are to be approximately equal, construct the schedule.

We have
$$\begin{aligned} R &= A_0 \cdot \frac{1}{a_{\overline{10}|}} \text{ at } 4\% \\ &= \$60{,}000 \times 0.1232909 \\ &= \$7397.45. \end{aligned}$$

In order to have an exact number of bonds, and in view of the fact that the interest on $60,000 for 1 yr. at 4% is $2400, we let $R = \$7400$.

YEAR	PRINCIPAL, BEGINNING OF YEAR	INTEREST FOR THE YEAR	ANNUAL PAYMENTS	PRINCIPAL REPAID	NUMBER OF BONDS RETIRED
21	60,000	2,400	7,400	5,000	5
22	55,000	2,200	7,200	5,000	5
23	50,000	2,000	7,000	5,000	5
24	45,000	1,800	7,800	6,000	6
25	39,000	1,560	7,560	6,000	6
26	33,000	1,320	7,320	6,000	6
27	27,000	1,080	7,080	6,000	6
28	21,000	840	7,840	7,000	7
29	14,000	560	7,560	7,000	7
30	7,000	280	7,280	7,000	7
Totals	351,000	14,040	74,040	60,000	60

Exercise 65. Amortization of Serial Bonds

1. A city borrows \$200,000 at 5%, to be repaid in 20 equal annual installments. Find the annual payment.

2. In Ex. 1, if the \$200,000 is borrowed on 200 bonds of \$1000 each and the repayment is to be made in 20 yr. in sums as nearly equal as possible, construct the schedule.

3. A manufacturing company arranged to issue 20 six-per-cent bonds of \$100 each. It desired to fix the times of maturity so that the first bonds maturing should be paid at the end of 11 yr. and additional bonds each year thereafter for 4 yr., and wished to make these five annual charges for interest and the face of the bonds as nearly equal as possible. Find the number of bonds that should mature each year.

4. In Ex. 3 construct the amortization schedule for the bond issue.

5. Construct the schedule for the repayment of \$5000 in \$100 bonds, interest at 6%, if the annual payments are to be as nearly equal as possible, the repayment to be made in 5 yr.

6. If the principal in Ex. 5 were to be repaid in 6 yr., how many bonds would be retired each year, keeping the total annual payments as nearly equal as possible?

7. For the purpose of erecting a school building, a board of education issues \$60,000 worth of bonds of \$1000 each. The bonds bear 5% interest and are to be such that some will mature in 15 yr., some in 16 yr., and so on, the last of the series maturing at the end of 20 yr. It is desired to have the total payments for principal and interest of these bonds during the six maturing years as nearly equal as possible. How many bonds should be paid each year?

8. Construct the schedule for Ex. 1 to 5 yr. only, using the exact sum thus found as the annual payment.

144. Finding Time for Payment of Debt. Since n enters as an exponent in the formula for A, it is necessary to use logarithms in finding its value. Four examples will serve to illustrate the process.

1. A corporation estimates that it can spare \$360 annually from its net income for the purpose of financing a loan of \$3000 at 6%, principal and interest to be paid in equal annual installments. How long will it take to pay the debt?

Since, by § 131, $$A_0 = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right],$$

we have $$3000 = \frac{360}{0.06}\left(1 - \frac{1}{1.06^n}\right);$$

whence $$\frac{1}{2} = 1 - \frac{1}{1.06^n},$$

and $$1.06^n = 2.$$

Therefore $$n = \frac{\log 2}{\log 1.06} = \frac{0.301030}{0.025306} = 11.8956.$$

That is, the payments are to be made annually for 11 yr., after which the balance may run 0.8956 yr., or approximately 322 days, when the interest, together with the outstanding principal, will be equal to 0.8956 of the payment for a year.

The amortization schedules for this example and the one following are given on the opposite page.

The student should consider the reason why the second check suggested in § 141 does not apply to this case.

2. In Ex. 1, if the corporation decided to make the time exactly 12 yr., how much would it have to pay annually in order to liquidate the debt?

In this case we have to find R instead of n.

Hence we have
$$R = A_0 \cdot \frac{1}{a_{\overline{12|}}} \text{ at } 6\%$$
$$= \$3000 \times 0.1192770.$$

Therefore the annual payment would be \$357.83.

AMORTIZATION SCHEDULE FOR § 144, EX. 1

Year	Principal, Beginning of Year		Interest for the Year		Annual Payments		Principal Repaid	
1	3000		180		360		180	
2	2820		169	20	360		190	80
3	2629	20	157	75	360		202	25
4	2426	95	145	62	360		214	38
5	2212	57	132	75	360		227	25
6	1985	32	119	12	360		240	88
7	1744	44	104	67	360		255	33
8	1489	11	89	35	360		270	65
9	1218	46	73	11	360		286	89
10	931	57	55	89	360		304	11
11	627	46	37	65	360		322	35
12	305	11	16	37	321	48	305	11
Totals	21,390	19	1281	48	4281	48	3000	

AMORTIZATION SCHEDULE FOR § 144, EX. 2

Year	Principal, Beginning of Year		Interest for the Year		Annual Payments		Principal Repaid	
1	3000		180		357	83	177	83
2	2822	17	169	33	357	83	188	50
3	2633	67	158	02	357	83	199	81
4	2433	86	146	03	357	83	211	80
5	2222	06	133	32	357	83	224	51
6	1997	55	119	85	357	83	237	98
7	1759	57	105	57	357	83	252	26
8	1507	31	90	44	357	83	267	39
9	1239	92	74	40	357	83	283	43
10	956	49	57	39	357	83	300	44
11	656	05	39	36	357	83	318	47
12	337	58	20	25	357	83	337	58
Totals	21,566	23	1293	96	4293	96	3000	

3. If I buy a phonograph for \$100, paying \$10 cash and agreeing to pay the rest in monthly installments of \$8, each of which is to cover the interest then due at 6% per annum and to reduce the balance due on the principal, how many months should the installments continue?

We have a series of ordinary annuity payments of \$8 per period, the interest being 0.005 per period, and the present value being \$90.

Since, by § 131, $A_0 = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right],$

we have $90 = \frac{8}{0.005}\left(1 - \frac{1}{1.005^n}\right);$

whence $0.05625 = 1 - \frac{1}{1.005^n},$

or $\frac{1}{1.005^n} = 0.94375.$

By logarithms,

$$n = -\frac{\log 0.94375}{\log 1.005}$$

$$= \frac{0.025143}{0.002166}$$

$$= 11.608, \text{ the number of months.}$$

Hence there will be 11 monthly installments of \$8 each, and a twelfth installment of 0.608 of \$8, or \$4.86; this in addition to the \$10 down.

4. Owing to increased cost of manufacture a dealer decides that the sale price of a certain line of pianos must be raised to \$550 net. He allows the option, however, of paying \$550 cash, or of paying \$50 down and the rest in monthly payments of \$10 each until the debt with interest on unpaid balances at 6% is extinguished. How many months must the payments of \$10 each continue?

As above, we have $500 = \frac{10}{0.005}\left(1 - \frac{1}{1.005^n}\right).$

Solving for n, we find that $n = 57.68$.
This requires 58 payments of \$10 each, in addition to \$50 down.

Exercise 66. Finding the Number of Payments

1. Find the time in which a debt of $2000 with interest at 6% can be paid in annual installments of $800 each.

2. In how many years can a $100,000 bond issue bearing 5% interest be amortized if the conditions of issue stipulate an annual installment of $10,000?

3. How soon will a debt of $8500 bearing 4% interest be extinguished by annual installments of $850 each?

4. In Ex. 3 how long will it take to pay the $8500 if installments of $425 are paid semiannually and interest is due semiannually?

5. A borrows $3000 at 6% interest and agrees to repay it in annual payments of $500. How long will it take?

It is customary in such a case to say that the loan bears interest at 6% with 6% for amortization.

6. A company borrowed $10,000, agreeing to pay 6% interest and 6% for amortization each year. In how many years will the obligation be satisfied?

7. If a railroad company can spare $15,000 a year from its net income to finance a loan of $100,000 bearing 5%, how many years will it take to satisfy the loan?

8. An installment house found it necessary to revise the terms of sale on several articles because of an increase in the cost. The monthly installments on articles selling for less than $100 were $5; on those whose price was between $100 and $500, $10 a month; on those sold at prices between $500 and $1000, $15 a month. In each case the first installment was paid in advance. If the house charges interest at the rate of 6% per annum, find the nearest integral number of months of payment for articles which sell for $85, $250, $800, and $900 respectively.

145. Sinking Fund. The amortization funds thus far considered were used to pay the interest as it falls due and also to reduce the principal periodically. From the creditor's standpoint this is not the most desirable arrangement, for he must then reinvest each year the part of the principal repaid. It is therefore more usual to have the entire principal repaid at the end of a stipulated time, the interest being paid annually. In this case the debtor may set aside equal amounts annually, investing them at current rates, and may thus prepare to meet the maturing obligation.

A fund accumulated for the purpose of paying the full debt when due is called a *sinking fund.*

146. Illustrative Example. Lacking \$3000 of the purchase price of a farm, a man borrows this sum on a bond and mortgage, the bond to be repaid in 5 yr. He decides to deposit at the end of each year in a savings bank paying 4%, compounded annually, such a sum as shall enable him to liquidate the debt in 5 yr. How much must he deposit?

We evidently have an ordinary annuity of five terms, the final value being \$3000 and R being unknown.

Since, by § 126, $A_n = Rs_{\overline{n|}}$,

we have $\$3000 = Rs_{\overline{5|}}$ at 4%;

whence
$$R = \$3000 \cdot \frac{1}{s_{\overline{5|}}} \text{ at } 4\%$$
$$= \$3000 \times 0.1846271$$
$$= \$553.88.$$

Since 5 × \$553.88 = \$2769.40, the man must save this amount, besides the annual interest charge, in order to pay the \$3000, the balance of \$230.60 being realized in interest.

If he had been able to realize 6% instead of 4%, we should have

$$\$3000 = Rs_{\overline{5|}} \text{ at } 6\%,$$

and then $R = \$532.19$,

the total saving on account of principal being 5 × \$532.19, or \$2660.95.

147. Interest Rate and Sinking-Fund Rate Different. In the example of § 146 we found that $553.88 per annum would provide for the payment of the principal when due, but that in addition to this the interest must also be paid. This interest may be at a higher rate than that realized on the payments into the sinking fund. If the rate of interest were 5% on the debt, the annual interest charge would be $150, making the total annual charge $703.88. The total expenditure would then be 5 × $703.88, or $3519.40.

148. Schedule of Expenditure. In a case like that of § 147 a schedule showing the amortization of the debt and the expenditure for interest may be prepared as follows:

YEAR	SINKING FUND, BEGINNING OF THE YEAR		INTEREST ON SINKING FUND		ANNUAL PAYMENTS		ADDED TO SINKING FUND		INTEREST ON DEBT		ANNUAL CHARGE	
1					553	88	553	88	150		703	88
2	553	88	22	16	553	88	576	04	150		703	88
3	1129	92	45	20	553	88	599	08	150		703	88
4	1729		69	16	553	88	623	04	150		703	88
5	2352	04	94	08	553	88	647	96	150		703	88
Totals	5764	84	230	60	2769	40	3000		750		3519	40

The total of the sinking fund must equal the debt, and this affords one check. Similarly, the sum of the annual payments ($2769.40) and the interest on the sinking fund ($230.60) must also equal the debt. The student should find two other checks of a similar nature, based upon the above schedule.

The importance of being able to calculate the annual charge necessary to provide for the interest and sinking fund in municipal finance is apparent. The laws of most states require the creation of a sinking fund to retire every bond issue. The importance of the subject is also apparent in the finance of corporations. The amortization schedule should be worked out in detail before the bonds are issued.

149. Interest Rate and Sinking-Fund Rate Identical. If the interest rate is the same as the sinking-fund rate, a noteworthy relation exists between the sum raised for the interest and the sinking fund and that raised for the amortization of the debt in equal annual installments.

If D represents the face of the debt, i the rate of interest on the debt and also on the sinking fund, R the sinking-fund annuity, and C the total annual charge, then

$$\text{Interest on } D = Di,$$

and
$$R = \frac{D}{s_{\overline{n}|}}$$
$$= D \cdot \frac{i}{(1+i)^n - 1}.$$

Also
$$C = Di + D \cdot \frac{i}{(1+i)^n - 1};$$

whence
$$C = Di \cdot \frac{(1+i)^n}{(1+i)^n - 1}.$$

If the debt is amortized by equal annual payments to cover interest and the reduction of the principal, D is the present value of the annuity whose annual rent, R, is to be found.

Hence
$$D = Ra_{\overline{n}|},$$

and so
$$R = D \cdot \frac{1}{a_{\overline{n}|}}.$$

Expanding $\frac{1}{a_{\overline{n}|}}$,
$$R = Di \cdot \frac{(1+i)^n}{(1+i)^n - 1}.$$

Hence
$$C = R.$$

That is, *when the accumulation rate for the sinking fund is the same as the rate of interest paid on the debt, the amount needed for interest and sinking fund is equal to the amount needed to pay the principal and interest in the same number of equal annual installments.*

150. Illustrative Examples. 1. A town borrows \$150,000, to be repaid at the end of 20 yr. with interest at $4\frac{1}{2}\%$, and agrees to establish a sinking fund for the repayment of the principal when due. If the sinking fund earns 4% compound interest, find the total annual charge.

The annual interest is $4\frac{1}{2}\%$ of \$150,000, or \$6750.

For the sinking fund, $R = \$150{,}000 \cdot \dfrac{1}{s_{\overline{20|}}}$ at $4\% = \$5037.26$,

and $$C = \$6750 + \$5037.26 = \$11{,}787.26.$$

2. In Ex. 1, given that the rate is $4\frac{1}{2}\%$ for both the interest and the sinking fund, find the total annual charge.

We now have
$$R = \$150{,}000 \cdot \frac{1}{s_{\overline{20|}}} \text{ at } 4\tfrac{1}{2}\%$$
$$= \$150{,}000 \times 0.0318761 = \$4781.42.$$

Adding \$6750, as in Ex. 1,
$$C = \$6750 + \$4781.42 = \$11{,}531.42.$$

If we prefer, we may use the amortization formula,
$$\$150{,}000 = Ra_{\overline{20|}} \text{ at } 4\tfrac{1}{2}\%.$$

Then
$$R = \$150{,}000 \cdot \frac{1}{a_{\overline{20|}}} \text{ at } 4\tfrac{1}{2}\%$$
$$= \$150{,}000 \times 0.0768761 = \$11{,}531.42.$$

3. A corporation can spare \$30,000 annually to finance a bond issue, including interest at 5% on the bonds and also including the charges for the sinking fund. If the sinking fund earns 4% compound interest, find the number of 20-year bonds of \$100 each that it can issue.

If A_n is the face of the bond issue, then $0.05\,A_n$ is the annual interest charge and $0.0335817 \cdot A_n$ is the annual sinking-fund charge.

Hence $$0.0835817 \cdot A_n = \$30{,}000,$$

and $$A_n = \$358{,}930.24.$$

Hence the corporation can float an issue of 3589 bonds at a total annual cost of \$17,945 for interest and \$12,052.47 for the sinking fund.

Exercise 67. Annual Sinking-Fund Charge

1. A city borrows \$100,000 at $4\frac{1}{2}\%$ and establishes a sinking fund, accumulating at 3% compound interest, to pay the debt at the end of 25 yr. Find the annual payment to be made into the sinking fund.

2. In Ex. 1 find the sum to be raised annually to pay the interest and sinking-fund charges.

3. In Ex. 1, if the city could accumulate its sinking fund at $4\frac{1}{2}\%$ compound interest, what sum would be needed annually to meet the interest and provide for the sinking fund?

4. A town borrows \$300,000 at 5% on bonds maturing in 20 yr. and accumulates a sinking fund at 4% compound interest. How much would it have gained or lost if it had borrowed at $5\frac{1}{2}\%$ and had accepted the privilege of repaying principal and interest in 20 equal annual installments?

5. A corporation calculated that it could spare \$100,000 annually out of its net income to float a bond issue at 5%, redeemable in 20 yr., and to provide a sinking fund realizing 4% compound interest. If the bonds were in denominations of \$100 each, find the number of bonds issued.

6. Construct a schedule of expenditure for the bond issue of Ex. 5 for the first 5 yr.

7. A corporation calculated that it could spare \$250,000 annually to finance a bond issue, including interest at $2\frac{1}{2}\%$ semiannually on the bonds and also including the charges for the sinking fund. If the sinking fund earns 4% interest, compounded semiannually, find the number of 25-year bonds of \$1000 each that the company can issue.

8. What annual sinking-fund payment will provide for the payment of \$800,000 due in 30 yr., the sinking fund accumulating at 4% compound interest?

Exercise 68. Review of Chapter VIII

1. Having given a mortgage on his house for $2400, Mr. S made a contract with the mortgagee for its extension for 3 yr., with the privilege of paying any multiple of $50 on the principal on any interest day. The mortgage bears 6% payable semiannually. What semiannual payments, covering interest and payment on principal, must he make to extinguish the mortgage in the 3 yr., the yearly burdens being as nearly equal as possible?

2. A corporation borrows $300,000 on bonds bearing 5%, due in 15 yr. The bonds stipulate that a sinking fund shall be created by the corporation, into which fund there shall be paid a sufficient amount annually to amortize the principal when due, on a 4% basis. Find the annual payment to the sinking fund and the annual interest charge.

3. In Ex. 2 construct the amortization schedule showing the accumulation of the sinking fund.

4. A corporation borrowed $500,000 on 20-year bonds, establishing a sinking fund to amortize the debt on a 4% basis after 6 yr. Find the annual payment to the sinking fund.

5. In Ex. 4 construct the schedule showing the amortization of the debt.

6. How long will it take to pay the principal and interest of a debt of $10,000 at 5% in equal annual installments of $1000 each?

7. Wishing to borrow $85,000, a municipality issued bonds of $500 each, bearing interest at 5%. It proposed to equalize as nearly as possible the annual burden for interest paid and principal retired. Find the annual charge necessary to make the burdens equal and the nearest integral number of bonds retired during the first 5 yr.

Exercise 69. Review of Chapters I–VIII

1. Find by the Binomial Theorem the compound amount of 1 for 3 yr. at 4%.

2. Expand $(1+i)^5$, and use the result to find the compound amount of 1 for 5 yr. at 3%.

3. Find by logarithms the compound amount of $8500 for 9 yr. at $3\frac{1}{4}$%.

4. A man purchased a piano for $800, paying $200 down and $20 a month with interest on the unpaid balances at 6% per annum. Find the sum of the monthly payments.

5. Prove that the logarithm of the product of three numbers is equal to the sum of the logarithms of the numbers.

6. B owes C two debts, one of $800 due in 3 yr., and one of $1200 due in 5 yr., interest at 6% in each case. He desires to commute these debts into two equal sums, one payable in 2 yr., and the other in 4 yr. Find the commuted sums.

7. How much should I pay for the right to receive an annuity of $500 a year for 15 yr., if money is worth 5%?

8. In Ex. 7 how much should I set aside annually to accumulate the amount necessary to purchase the annuity in 10 yr., the sums set aside bearing 5% compound interest?

9. A man owns a certain number of shares in an electric company which declares semiannual dividends of 4%. If he bought the shares at $100 each, what rate is he realizing per annum on his investment?

10. An investor who desired to realize 6% per annum on his money was offered stock at $102 a share which had been paying semiannual dividends of $3 a share. By how much would the effective rate on this investment differ from the desired rate, assuming that the shares remain at $102 and the dividends remain constant?

CHAPTER IX

DEPRECIATION

151. Consumption Goods and Capital Goods. Economists distinguish between *consumption goods*, or goods that are used up in directly satisfying human wants, and *capital goods*, or goods that are used to produce other goods. Capital goods do not directly satisfy wants. They represent the savings of society, or the "congealed labor," dedicated to further production.

Thus lead pencils are consumption goods, since they are used in satisfying the want of writing. The planing and turning machines used in producing the lead pencils are capital goods, or capital. The factory in which the pencils are made is a capital good, but a dwelling house is a consumption good. The line of demarcation between these classes is not always sharply defined.

152. Investment. The money paid for capital goods needed to conduct a business is called the *investment* in that business. The term is also applied to the money paid for bonds or other instruments that yield a money return to their possessor. If time and effort are spent in producing a capital good, they may be said to be *invested* in that good.

All investment represents social saving. If a man saves more money than he needs for comfortable living, he has a margin for investment. This investment may be made, for example, in municipal bonds, thus furnishing capital needed to conduct the business of the municipality; it may go into the capital of a business corporation; or it may be lent to a private individual. In each of these cases the investment represents a social saving.

153. Cost of Production. The money value of the raw material, labor, managing ability, wear and tear on capital goods, and other costs necessarily associated with the production of a unit of a product is called the *cost of production* of that unit. The price at which such unit sells must cover all these costs if the business of the house is to continue.

For example, if a machine costing $100 can produce 1000 articles before it becomes worthless, and the raw material, labor, etc. involved in the production of these articles cost $400, the total cost of producing the 1000 articles is $500, or 50¢ each.

154. Depreciation. The lessening in value of buildings, machinery, or other capital goods through use which cannot be made good by current repairs is called *depreciation*. In order to prevent a loss the income from the sale of the product must cover this item.

When capital is invested in a productive enterprise it should be kept intact, so that the plant or physical assets plus the funds or securities reserved for the purpose may together be equal to the capital invested. A plant is also kept valuable, and is safeguarded by current repairs made out of funds taken from profits or earnings and by insurance against fire, tornado, or other catastrophe.

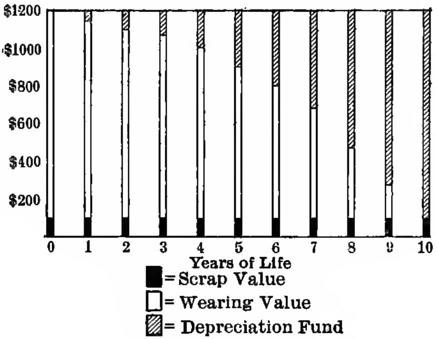
If, however, machinery or buildings are used for a number of years, even if current repairs are made, the value of these assets becomes less with use. An engine, a boiler, or a loom will wear out, even though repairs are made and the machinery is overhauled periodically. If no provision is made for the replacing of such worn parts out of the earnings or profits, the capital invested has been impaired to the extent of the loss of value of the worn machines.

The recent tendency of state legislation in this country is to insist upon adequate provision for the item of the depreciation of the property by all public-service corporations and all corporations enjoying natural monopolies. It is evident that if additional capital is borrowed from time to time by such corporations to replace worn parts or worn capital goods of any kind, the charge for this purpose will eventually become too excessive to be carried.

155. Depreciation Reserves and Depreciation Funds. Sufficient funds should be set aside annually out of the net income to offset the loss of value of the capital goods before any dividends are paid. If such funds are used to replace or to supplement capital goods or are otherwise put back into the business, these funds constitute a *depreciation reserve.*

If, however, the money thus set aside is kept separate and is accumulated until some part of the capital goods must be replaced, that money constitutes a *depreciation fund.*

Until recently but little attention was given to the consideration of depreciation reserves and depreciation funds, even by large corporations. Of late, however, various public-service commissions have required the corporations under their control to establish such reserves or funds and to "write off" from the value of their assets each year to allow for so doing. New economic demands, such as those imposed by the war in Europe and also by the consequent government control, have all required a much closer scrutiny of business.



If the exact amount of depreciation of an asset could be determined periodically, and if the depreciation fund could be brought to such an amount that the remaining wearing value plus the scrap value and the depreciation fund would be exactly the same as the cost of new material, the capital would be kept absolutely unimpaired. The history of a machine and its depreciation fund might then be as shown in the above graph.

156. Physical Depreciation and Functional Depreciation. The depreciation in the value of capital goods caused by wear and tear due to use is called *physical depreciation.*

The depreciation in the value of capital goods due to such causes as new inventions which tend to render machinery obsolete is called *functional depreciation.*

In the case of each of these types of depreciation, value must be *written off* periodically in the accounts which deal with the capital goods concerned.

The time that elapses before an asset is reduced to scrap value is called the *life of the asset.*

The difference between the cost of any new capital goods and their scrap value is called their *wearing value.*

157. Depreciation Allowance. Since the probable life of an asset must be estimated in advance and experience gives only an approximation to the actual life, it becomes necessary to provide a factor of safety known as a *depreciation allowance* or *depreciation-fund reserve.*

If this is not done, replacements may have to be made with borrowed money or by greatly reducing the dividends for the year in which they are made. Experience tables estimating the probable life of buildings, machines, and other assets are used to some extent.

The depreciation allowance set aside annually need not be exactly equal to the actual loss in value of any part of the capital goods for one year; it is merely essential that it is sufficient, together with any scrap value that may exist, to replace such part when necessary. Theoretical depreciation and practical depreciation need not keep pace with each other, but they should terminate at the same time.

Evidently it is desirable that the accounts of a business should itemize the capital goods at their approximate present value at the time of the inventory.

158. Straight-Line Method in Depreciation. The method of determining theoretical depreciation which assumes that if the life of a part, as of a machine, is n years, $1/n$th of the wearing value of the part must be set aside annually is called the *straight-line method.* In this method no interest is allowed on the sums set aside.

If D is the annual depreciation charge, n the number of years or periods in the life of the part, V the value of a new part, and V' the salvage or scrap value, then by the above definition

$$D = (V - V')/n,$$

and if V' is zero, the formula becomes

$$D = V/n.$$

159. Illustrative Example. A machine costing $1500 has a life of 10 yr. and a scrap value of $50. Find the annual depreciation allowance.

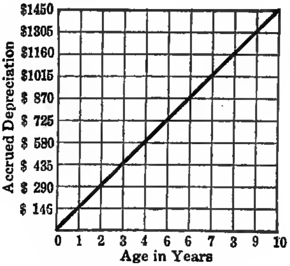
Here $V = \$1500$, $V' = \$50$, and $n = 10$. Therefore

$$D = \frac{\$1500 - \$50}{10} = \$145.$$

The facts may be set forth in a schedule, thus:

AGE IN YEARS	VALUE, END OF YEAR	DEPRECIATION ALLOWANCE	ACCRUED DEPRECIATION
1	1355	145	145
2	1210	145	290
3	1065	145	435
4	920	145	580
5	775	145	725
6	630	145	870
7	485	145	1015
8	340	145	1160
9	195	145	1305
10	50	145	1450

160. Graph of Depreciation. If we use the first and the fourth columns of the schedule in § 159 as variables, the resulting graph is a straight line, as here shown. This fact gives the method of reckoning its name, the *straight-line method*. It is extremely simple, and is generally used in small establishments. The methods given in §§ 161–168 are used by accountants and corporations whose accounts are kept scientifically.



Exercise 70. Straight-Line Method

1. The life of a building that cost $45,000 is 40 yr., and the scrap value $500. Find the annual depreciation charge.

2. A plant consists of the parts stated below. Fill the spaces and find the annual depreciation charge.

PART	LIFE	COST NEW	SCRAP VALUE	WEARING VALUE	ANNUAL DEP. CHARGE
Building	45 years	$100,000	$5,000		
Engine	20 years	8,500	300		
Boiler	15 years	2,500	200		
Looms	10 years	85,000	3,500		
Totals					

3. Make a table of computations for the depreciation account for the looms in Ex. 2.

4. A machine costing new $5600 has an estimated life of 15 yr. and a scrap value of $50. Find the annual depreciation charge by the straight-line method.

161. Reducing-Balance Method. Instead of charging off an equal amount from the value of a depreciating investment each year, certain accountants and certain corporations prefer to charge off a constant per cent of the reducing balance left after the previous year's charge. This method of providing a depreciation allowance is called the *reducing-balance method.*

The constant per cent must be so calculated that at the end of the estimated life of the asset the wearing value will have been written off.

For example, if a machine costing \$100 is to be reduced to value \$51.20 in a lifetime of 3 yr., the constant percentage can easily be shown to be 20. For at the end of 1 yr. the value would then be \$100 less 20% of \$100, or 80% of \$100, or \$80; at the end of 2 yr. it is 80% of \$80, or \$64; and at the end of 3 yr. it is 80% of \$64, or \$51.20. That is, the value at the end of the third year is 80% of 80% of 80% of \$100, which is the same as $0.80^3 \times \$100$.

162. Formula for Reducing Balance. The calculation of the constant per cent to be charged off in the reducing-balance method is most easily effected by logarithms.

Let V be the original value of the asset, V' the salvage or scrap value, n the number of years or periods before the asset is reduced to the salvage value, and x the constant per cent of the reducing balance remaining after writing off each time. Then it is evident that

$$Vx^n = V';$$

whence

$$x = \sqrt[n]{\frac{V'}{V}},$$

and

$$\log x = \frac{1}{n}(\log V' - \log V).$$

The constant per cent charged off is $1 - x$.

Having determined the constant per cent of the reduced balance remaining after writing off each time, the accountant is in position to construct his schedule.

163. Illustrative Example. A dynamo costing \$5000 has an estimated life of 10 yr. and a scrap value of \$200. Find the constant rate for the reducing balance and construct a schedule for the depreciation account.

Since $V = \$5000$, $V' = \$200$, and $n = 10$, we have

$$5000\,x^{10} = 200;$$

whence

$$x = \sqrt[10]{\tfrac{200}{5000}}$$

$$= \sqrt[10]{\tfrac{1}{25}},$$

and

$$\log x = \tfrac{1}{10}(\log 1 - \log 25)$$

$$= \tfrac{1}{10}(0 - 1.397940)$$

$$= -\,0.139794$$

$$= 9.860206 - 10.$$

Therefore

$$x = 0.72478,$$

and

$$1 - x = 0.27522, \text{ the required rate.}$$

The schedule, commonly called the *table of computations*, can now be constructed as follows:

AGE IN YEARS	REDUCED BALANCE		DEPRECIATION CHARGE		ACCRUED DEPRECIATION	
1	3623	90	1376	10	1376	10
2	2626	53	997	37	2373	47
3	1903	66	722	87	3096	34
4	1379	73	523	93	3620	27
5	1000		379	73	4000	
6	724	78	275	22	4275	22
7	525	31	199	47	4474	69
8	380	73	144	58	4619	27
9	275	95	104	78	4724	05
10	200		75	95	4800	

We see that the depreciation charges are heavy during the early years, but the current repairs will then be light. Later, when the repairs are heavier, the depreciation charges are greatly reduced, thus equalizing the burden. Furthermore, it should be recalled that the market value of a part decreases most rapidly during the first years.

Exercise 71. Reducing-Balance Method

1. A factory building costing $250,000 is estimated by the architect to have a probable life of 30 yr., and that the salvage for materials will be at least $8000. The owning corporation uses the reducing-balance method of writing off value of capital goods. Find the constant rate of deduction necessary to reduce the building value to salvage in 30 yr.

2. In Ex. 1 the factory contains 25 machine tools of a certain type costing $550 each, and their estimated life under constant day usage is 8 yr., the salvage value of each machine being $50. Find the reducing-balance rate of deduction necessary to reduce the machines to salvage value in 8 yr.

3. In Ex. 2 construct the table of computations for one of the machines.

4. After the machines of Ex. 2 had been in use for 2 yr. the growth of business necessitated night shifts. This halved the remaining probable life of the machines. Find the new reducing-balance rate of deduction for depreciation to bring the machines to salvage value at the end of the remaining 3 yr.

5. In Ex. 4 draw a graph of the accrued depreciation.

6. In a certain rolling mill a machine cost $8500 when new and has an estimated life of 8 yr. and a salvage value of $25. Find the constant annual rate of deduction necessary to reduce the value of the machine to salvage in 8 yr.

7. In Ex. 6 construct the table of computations.

8. A part which cost $4500 when new has an estimated life of 15 yr. and a salvage value of $20. Find the constant annual rate of deduction necessary to reduce the value of the part to salvage in 15 yr. Construct the table of computations covering the life of the part and draw the graph of the accrued depreciation.

164. Sinking-Fund Method. The sinking-fund method of providing for the replacement of a part that is worn out includes the setting aside of a fixed sum during each year or period and the investment of these sums at compound interest at a definite rate. These annual contributions, together with their interest accumulations, must be such as to amount to the wearing value of the part at the time it is scrapped. If these funds are reinvested in the business, the interest must be taken out of the net income annually, as must also the fixed annual contributions to the sinking fund.

165. Finding the Annual Charge. When we use the method of § 164, it is evident that we have an ordinary annuity of which we know not only the final value, being the wearing value, but also the rate of interest and the number of terms, and in which the unknown quantity is R, the annual rent.

Since, by § 126, $$A_n = Rs_{\overline{n}|},$$

we have $$R = A_n \cdot \frac{1}{s_{\overline{n}|}}.$$

The values of $1/s_{\overline{n}|}$ for the usual rates of interest may be found from the tables on pages 226–229. If the sinking fund is accumulated at a rate not found in the tables, R can be calculated by the use of logarithms. For this purpose we use the expanded form of the above formula, thus:

$$R = A_n \cdot \frac{i}{(1+i)^n - 1},$$

from which $$(1+i)^n = \frac{A_n}{R} i + 1,$$

and $$n = \frac{\log\left(\frac{A_n}{R} i + 1\right)}{\log(1+i)}.$$

166. Illustrative Examples. 1. A machine that cost \$5000 has an estimated life of 10 yr. and a salvage value of \$100. If the depreciation fund can be invested at 4%, what is the annual rate of depreciation?

Letting A_n be the wearing value, we have $A_n = \$4900$, $n = 10$, and $i = 4\%$.

Then
$$R = \$4900 \cdot \frac{1}{s_{\overline{10}|}} \text{ at } 4\%$$
$$= \$4900 \times 0.0832909$$
$$= \$408.1254.$$

Therefore the annual rate of depreciation is found as follows:
$$r = \$408.125 \div \$4900$$
$$= 8.33\%.$$

The table of computation is as follows:

AGE IN YEARS	VALUE, END OF YEAR		PAYMENT TO S. F.		INTEREST AT 4%		ANNUAL ADDITIONS		VALUE OF S. F.	
1	4591	87	408	13			408	13	408	13
2	4167	42	408	12	16	33	424	45	832	58
3	3725	99	408	13	33	30	441	43	1274	01
4	3266	91	408	12	50	96	459	08	1733	09
5	2789	46	408	13	69	32	477	45	2210	54
6	2292	92	408	12	88	42	496	54	2707	08
7	1776	51	408	13	108	28	516	41	3223	49
8	1239	45	408	12	128	94	537	06	3760	55
9	680	90	408	13	150	42	558	55	4319	10
10	100	02	408	12	172	76	580	88	4899	98
Totals			4081	25	818	73	4899	98		

If a plant consists of several parts having different probable lengths of life, the total payment to the depreciation fund in any year is the sum of the payments for the separate parts. The probable life, salvage value, and sinking-fund rate being known for each part, the annual payment to the depreciation fund is calculated for each part as above. This case is illustrated on page 156.

2. Assuming that the sinking fund can be accumulated on a 3% basis, find the annual depreciation charge on the following plant:

PART	LIFE	COST	SCRAP	WEARING VALUE	DEPRECIATION	
Building	40 yr.	20,000	1000	19,000	251	99
Engine	20 yr.	8,000	200	7,800	290	28
Looms	15 yr.	10,000	2000	8,000	430	13
Totals		38,000	3200	34,800	972	40

Calculating the annual depreciation charge for each item separately gives the results in the last column.

The annual depreciation rate on the entire wearing value of the plant is \$972.40 ÷ \$34,800, or 0.0279.

It is, of course, immaterial whether the funds are united or are kept separate. If they are united and are regarded as one fund, \$8000 can be withdrawn to replace the looms at the end of 15 yr., \$7800 to replace the engine at the end of 20 yr., and so on.

167. Composite Life of a Plant. The time required for the total of the annual depreciation charges of a plant to become equal to the total wearing value when accumulated at the assumed rate of interest is called the *composite life of the plant.*

Bonds secured by a mortgage on the plant should not be issued for as long a period as this.

Solving Ex. 2 of § 166 for the composite life, we have

$$A_n = \frac{R}{i}[(1+i)^n - 1];$$

whence
$$34{,}800 = \frac{972.40}{0.03}(1.03^n - 1).$$

Therefore
$$1.03^n - 1 = 1.0736,$$

and
$$n \log 1.03 = \log 2.0736;$$

whence
$$n = 24.7.$$

Therefore the composite life of the plant is 24.7 yr.

Exercise 72. Sinking-Fund Method

1. The equipment used in a certain business cost $10,000 and has an estimated life of 10 yr., with no scrap value. If 4% is allowed on the depreciation fund, what is the annual depreciation charge?

2. In Ex. 1 construct the table of computations for the sinking fund.

3. Find the wearing value and the annual depreciation charge on a 2% basis for a plant consisting of the following parts, thus completing the schedule as given:

PART	LIFE	COST	SCRAP	WEARING VALUE	DEPRECIATION
Building	40 yr.	35,000	5000		
Engine	20 yr.	8,000	500		
Dynamos	15 yr.	18,000	1200		
Wiring	30 yr.	10,000	2000		

4. The life and the wearing value of the several parts of the plant of a public-service corporation were estimated by a commission as here given. Complete the following schedule on a 2% basis for depreciation:

PART	LIFE	WEARING VALUE	DEPRECIATION	RATE PER YEAR
A	20 yr.	$5,320,000		
B	10 yr.	2,700,000		
C	15 yr.	1,520,000		
D	18 yr.	850,000		
E	12 yr.	90,000		
F	10 yr.	1,250,000		

5. Find the composite life of the plants in Exs. 3 and 4.

168. Annuity Method. There is also a method of finding the depreciation which aims at finding such a sum as, withdrawn annually from the net income, shall allow a given rate of interest on the remaining value of the investment, the balance of each annual withdrawal constituting the depreciation allowance. The total of these withdrawals must, of course, reduce the value of the part considered to salvage value or to zero at the end of its life. This method is called the *annuity method.*

For example, if a part is worth $100 and has a life of 3 yr. with scrap value zero, it may be found that there will be required, by the method explained on page 159, an annuity of $36.721 to cover the interest at 5% and the depreciation allowance, as may be seen from the following table:

AGE IN YEARS	VALUE, END OF YEAR		INTEREST ON REMAINDER		INTEREST AND DEPRECIATION		DEPRECIATION ALLOWANCE	
1	68	279	5		36	721	31	721
2	34	972	3	414	36	721	33	307
3	0		1	749	36	721	34	972
Total of depreciation allowance = original value =							100	

It may be seen from the table that the interest at 5% for the first year on the remaining value of the part is estimated on $100, the value of the part at the beginning of the year. The value of the machine is reduced $36.721 and is therefore $68.279. Interest on $68.279 at 5% is $3.414, and this leaves the depreciation allowance $33.307.

This method allows interest only on capital actually in active use, the interest allowances on a given part decreasing as the depreciation allowances increase.

If the money invested in a capital good was originally borrowed, the method above shown allows for the payment of interest in full at 5%, provided the depreciation allowances also bear interest at 5%. If this is the case, the interest burden is carried, and the principal is kept intact for reinvestment in a new machine or part at the end of the 3 yr.

169. Calculation of Annuity. We found in § 168 that we practically were required to calculate an annuity. We shall now consider the method of this calculation, showing how, in § 168, the annuity $36.721 was found.

Let V be the original value, V' the salvage value, i the interest rate, and x the annuity.

Then the value at the end of 1 yr. is $V(1+i)-x$.

The value at the end of 2 yr. can evidently be found by multiplying this $V(1+i)-x$ by $1+i$ and then subtracting x as before. This gives for the value at the end of 2 yr.

$$[V(1+i)-x](1+i)-x.$$

Similarly, the value at the end of 3 yr. is

$$\{[V(1+i)-x](1+i)-x\}(1+i)-x,$$

which, by the given conditions as stated in § 168, is the scrap value V'. Removing parentheses, we have

$$V(1+i)^3-x(1+i)^2-x(1+i)-x=V'.$$

If the number of years were n, we should have

$$V(1+i)^n-x(1+i)^{n-1}-x(1+i)^{n-2}-\cdots-x(1+i)-x=V'.$$

By transposing and rearranging, we have

$$x[(1+i)^{n-1}+(1+i)^{n-2}+\cdots+(1+i)+1]=V(1+i)^n-V'.$$

Summing the geometric progression in the brackets,

$$\frac{(1+i)^n-1}{(1+i)-1}\cdot x=V(1+i)^n-V'.$$

Therefore $$\boldsymbol{x=[V(1+i)^n-V']\cdot\frac{i}{(1+i)^n-1}}.$$

Applying this to the problem in § 168, we have

$$x=(100\times 1.05^3-0)\cdot\frac{0.05}{1.05^3-1}$$
$$=36.721.$$

That is, the annuity is $36.721.

Exercise 73. Annuity Method

1. A plant worth \$50,000 has a life of 15 yr. and a salvage value \$1000. Find the annuity sufficient to cover interest on the remaining value of the investment at 5% and provide a depreciation fund.

2. In Ex. 1 construct the table of computations of the annuity and interest allowances.

3. Supposing that the salvage value in Ex. 1 is zero, calculate the annuity.

4. Supposing that the salvage value in Ex. 1 is \$500, calculate the annuity.

5. Supposing that the life of the plant in Ex. 1 is 30 yr. and the salvage value \$500, calculate the annuity.

6. A machine worth \$8000 has an estimated life of 11 yr. and a salvage value of \$200. The owner wishes to allow interest at 6% on the remaining values and provide depreciation allowances to reduce the value to salvage in the life period of the machine. Find the annuity.

7. Find the annuity in Ex. 6 if salvage value were zero.

8. In Ex. 6 construct the table of computations of the annuity and interest allowances.

9. In Ex. 6 calculate the annuity if the life of the machine were 20 yr. and the salvage value \$100.

10. A transformer cost \$5000 and has an estimated life of 15 yr. and a salvage value of \$50. The owner wishes to allow interest at 5% on the remaining values and provide depreciation allowances to reduce the value to salvage in the life period of the machine. Find the annuity.

11. In Ex. 10 construct the table of computations for the first 5 yr. of the life of the transformer and find the remaining value at the end of the fifth year.

Exercise 74. Review of Chapter IX

1. A company that manufactures typewriter ribbons has an inking machine that becomes worthless for its purpose after wear from 10 yr. to 12 yr. The original cost was $1400 and the salvage value is $100. The company wishes to reduce the machine to salvage value on their books in 10 yr. by the reducing-balance method. What per cent of the remaining value must be written off each year?

2. In Ex. 1 at what value will the machine be listed among the assets of the company at the end of the fifth year of its life?

3. In Ex. 1 construct the table of computations for the machine, showing its reduction to salvage value.

4. If the company mentioned in Ex. 1 had used the sinking-fund method of handling depreciation, realizing 4% on the fund, what annual contribution to the fund would be necessary from the net earnings?

5. In Ex. 4 construct the table of computations.

6. If the company in Ex. 1 had used the annuity method of reckoning depreciation allowance, what constant annual allowance would cover the depreciation and the interest at 5% on the remaining value of the investment?

7. Construct the schedule for Ex. 6.

8. By which of the above methods is the greatest value written off in the first 5 yr. of the life of the machine? By which method is the least value written off?

9. The machine mentioned in Ex. 1 requires repairs and overhauling as follows: first year, nothing; second year, $10; third year, $25; fourth year, $30; fifth year, $35; sixth year, $45; for the rest of its life, a total of $50. What method of reckoning depreciation should be adopted? State the reason.

Exercise 75. Review of Chapters I–IX

1. Find by logarithms the value of $\frac{1}{1.05^6} \cdot \$9000$.

2. Find by logarithms the value of $\frac{1}{1.0425^5}$, and use the result to find the present value of $45,000 payable in 5 yr., money being worth $4\frac{1}{4}\%$.

3. Make a rough drawing of a slide rule set to find the cost of any number of yards of cotton goods at 17¢ a yard.

4. By using a slide rule estimate the value of wheat worth $1.80 a bushel that will fill a bin 15 ft. long, 8 ft. wide, and 7 ft. deep, counting $1\frac{1}{4}$ cu. ft. to the bushel.

5. How long will it take $5000 to amount to $8500 at 5% compound interest?

6. How large a present debt can be canceled by paying to the creditor $600 at the end of each year for 4 yr., money being worth 5%?

7. At what rate per annum can a man afford to borrow money to pay a bill due in 60 da., with a discount from the face of the bill of 5% for immediate payment?

8. How large a debt can be amortized in 20 yr. by setting aside $5000 per annum and investing the funds at 4% per annum?

9. How large a debt can be financed by setting aside $5000 per annum to cover the interest at 5% and the annual contribution to the sinking fund necessary to amortize the debt at the end of 20 yr., the sinking fund earning 4% compounded annually?

10. In Ex. 9 how much more money could be borrowed if the interest rate were 4% and the sinking-fund rate were also 4%?

CHAPTER X

BOND VALUATION

170. Bond. An interest-bearing certificate of ownership of a debt, or a definite portion of a debt, issued under the seal of a government, a private corporation, or an individual, is called a *bond.*

The student should examine carefully the facsimile of a bond found on page 165. A bond is a promise to pay: (1) a principal sum or *face,* called also the *par value,* on or after a definite date, and (2) interest on the face at definite intervals and at a rate specified in the bond. The interval between interest payments is usually a half year or a year. Sometimes the bond provides that the principal may be repaid by the debtor and the bond canceled on any interest day after a specified date, or in some instances at the option of the issuing company after a notice published as provided in the bond. In the case of the cancellation at the option of the company there is often a stipulation in the bond that it shall be redeemed for a specified sum greater than its par value.

171. Kinds of Bonds. A bond the payment of which is secured by hypothecation of definite property in the shape of a mortgage or deed of trust, that is held by a trustee for the protection of the bondholders, is called a *mortgage bond.*

A bond that is not secured like a mortgage bond is called a *debenture bond* or simply a *debenture.*

In the case of debentures, only assets not specifically hypothecated are available for payment. Such bonds frequently contain a provision for a sinking fund to amortize the face when due. If issued by a municipality, the credit of the obligor is the sole pledge for their payment.

172. Coupon Bond. A bond having small notes known as *coupons* attached, these notes being promises to pay the interest as it becomes due, is called a *coupon bond.*

The coupons are usually numbered the same as the bond itself and are signed with the engraved signature of the treasurer of the issuing corporation. There is one coupon for each interest payment up to the date of maturity of the bond. Since the interest is usually payable semiannually, there are usually twice as many coupons as there are years to the date of maturity.

A coupon bond is shown on page 165.

173. Registered Bond. If the ownership of a bond is a matter of record on the books of the issuing corporation and the name of the owner is indorsed on the bond, the bond is called a *registered bond.*

If fully registered, both principal and interest are payable only to the registered holder or his legal representative, and the bond can be transferred to another person only by having the proper officer transfer it on the books of the company. If registered as to principal only, the interest coupons may be cashed by the bearer, but the principal will be paid only to the registered owner.

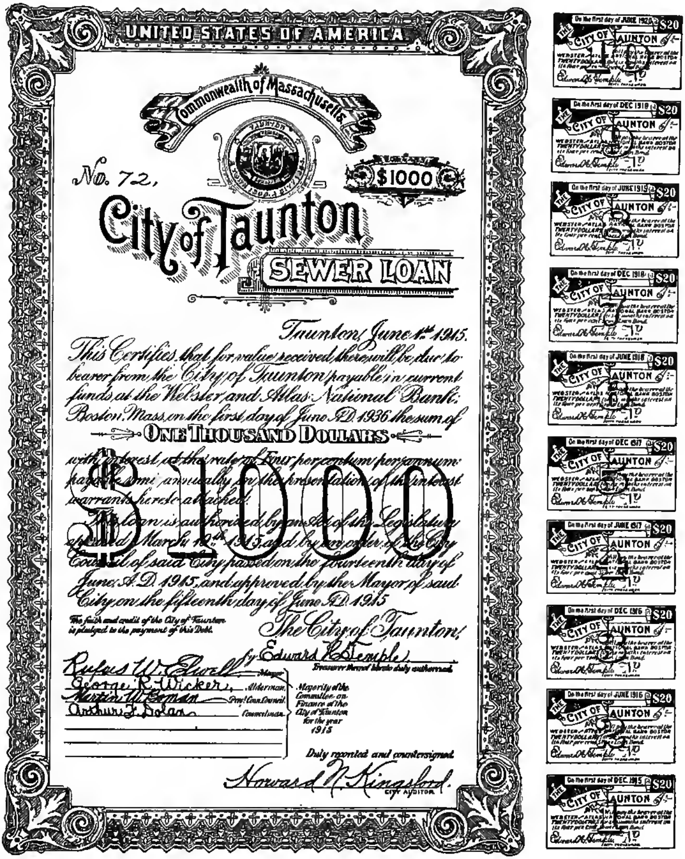
Coupon bonds often state that they may be registered on application by the holder. If a bond is registered as to principal only, the coupons remain attached; but if fully registered, a check for the interest when due is sent to the owner by mail.

174. Other Forms of Bonds. Bonds agreeing to pay the principal and interest in a definite number of equal installments are known as *installment bonds.*

Frequently it is provided that a certain number of bonds shall be called in by lot and paid annually after a certain time has elapsed.

British consols and French rentes have no date of maturity. They are perpetual loans, the owners receiving only the interest when due.

175. Illustration of a Coupon Bond. The following illustration shows a coupon bond issued by a municipality. Only a few of the coupons are here shown. On or after each interest day the proper coupon is cut off and cashed.



UNITED STATES OF AMERICA

Commonwealth of Massachusetts

No. 72,

$1000

City of Taunton

SEWER LOAN

Taunton, June 1st 1915.

This Certifies that for value received there will be due to bearer from the City of Taunton payable in current funds at the Webster and Atlas National Bank, Boston, Mass. on the first day of June A.D. 1936 the sum of

ONE THOUSAND DOLLARS

with interest at the rate of Four per centum per annum payable semi annually on the presentation of the interest warrants hereto attached.

This loan is authorized by an Act of the Legislature approved March 12th 1915, and by an order of the City Council of said City passed on the fourteenth day of June, A.D. 1915, and approved by the Mayor of said City on the fifteenth day of June A.D. 1915

$1000

The faith and credit of the City of Taunton is pledged to the payment of this Debt.

The City of Taunton,
by Edward H. Temple
Treasurer thereof thereto duly authorized.

Rufus W. Elwell — Mayor
George R. Wicker, — Alderman.
Martin W. Boman — Pres! Com. Council.
Arthur J. Dolan — Councilman.

Majority of the Committee on Finance of the City of Taunton for the year 1915

Duly recorded and countersigned
Howard N. Kingsford.
CITY AUDITOR.

On the first day of JUNE 1920 $20 THE CITY OF TAUNTON

On the first day of DEC 1919 $20 THE CITY OF TAUNTON

On the first day of JUNE 1919 $20 THE CITY OF TAUNTON

On the first day of DEC 1918 $20 THE CITY OF TAUNTON

On the first day of JUNE 1918 $20 THE CITY OF TAUNTON

On the first day of DEC 1917 $20 THE CITY OF TAUNTON

On the first day of JUNE 1917 $20 THE CITY OF TAUNTON

On the first day of DEC 1916 $20 THE CITY OF TAUNTON

On the first day of JUNE 1916 $20 THE CITY OF TAUNTON

On the first day of DEC 1915 $20 THE CITY OF TAUNTON

As with all municipal bonds, this is not secured by a mortgage. It is therefore, strictly speaking, a debenture bond.

176. Method of Designating Bonds. Bonds are usually designated by naming the issuing person or corporation, the interest rate, the time of maturity, and the initials of the months when the interest is payable. For example, "Penn. R.R. $4\frac{1}{2}$'s of '65, J. and D." indicates bonds of the Pennsylvania Railroad, bearing $4\frac{1}{2}\%$ interest, due in 1965, interest payable June 1 and Dec. 1 of each year.

Bonds issued by the United States often have special names, such as Liberty Bonds, Second Liberty Loan, Panama Canal Bonds, or 2's of 1950. In the same way a state bond may have some special name.

177. Valuation of a Bond. The value of a bond depends on several factors, as follows:

1. The security for the payment of principal and interest.
2. The price at which the bond will be redeemed by the obligor at any time.
3. The rate of interest specified.
4. The rate of interest to be realized on the investment.
5. The length of time before maturity.

Evidently the first of these items cannot always be closely estimated. Even an expert may be unable to tell within thousands of dollars the value of a plant mortgaged to secure a bond issue. In practice the strength and reputation of the issuing company or government may cause the investing public to pay such a premium as will bring the rate of interest realized below the market rate for money, this being due to the feeling of security in the investment.

178. Relation of Premium to Yield. If a bond is bought above par, less than the rate of interest specified in the bond will be realized; if it is bought below par, more than this rate will be realized.

Since only the face of a bond is payable at maturity, of course with the last interest charge, it is evident that any premium paid when the bond is purchased must, if the investment capital is to be kept intact, be saved out of the interest. The fewer the interest payments, the smaller the premium that can be made up out of these payments.

179. Illustrative Example. If I wish to buy on Jan. 1, 1920, a $1000 bond having 2 yr. to run and bearing 6% interest payable semiannually, and if I wish to realize 5% on the investment, what price can I afford to pay for the bond?

The interest received is $30 semiannually. If at the end of the 2 yr. I should receive for the bond all that I pay for it, I could afford to pay approximately $1200, because 2½% of $1200 is $30.

But at the end of the 2 yr. I receive only $1000, and thus my capital would be decreased $200. It can be shown, as on page 168, that if I pay $1018.81, I shall realize 2½% on my investment each half year and shall be able to save $18.80 out of the interest received to replace the premium I am obliged to pay.

Assuming for the moment that this is the case, the following schedule shows the process in detail:

Date	Interest on Book Value		Total Interest		Amortization Fund		Book Value	
1920, Jan. 1							1018	81
July 1	25	47	30		4	53	1014	28
1921, Jan. 1	25	36	30		4	64	1009	64
July 1	25	24	30		4	76	1004	88
1922, Jan. 1	25	12	30		4	88	1000	
Totals	101	19	120		18	81		

The interest on the book value at 2½% per half year is paid on a decreasing active investment beginning with a book value of $1018.81 and ending with $1004.88.

The difference between the 6% on the face and the 5% on the book value is set aside for amortization.

If the holder should sell the bond on Jan. 1, 1921, he should receive $1009.64, and the purchaser would then receive 5% on his investment.

While open to certain objections, this method is in universal use.

In this example $1018.81 is the *basis price* of the bond on Jan. 1, 1920.

The above bond is said to have been bought on a 5% basis on Jan. 1, 1920. In like manner bonds are said to be bought on a 4½% basis, a 5.2% basis, and so on, these rates being the rates of interest actually received on the money invested.

180. Separate Valuation of Principal and Annuity. One method of valuing a bond consists in taking the sum of the valuation of the principal and the valuation of the annuity. We shall first consider a special case, and in § 181 shall consider the general formula.

Find the value of a \$1000 bond bearing 6% interest, J. and J., due Jan. 1, 1928, bought Jan. 1, 1920, to yield 5%.

The receipts are \$1000, due in 8 yr., and an annuity of \$30 per period for 16 periods. Both of these items must be valued on the basis of $2\frac{1}{2}\%$ per period, the interest on the bond being convertible as received. We must therefore pay for the bond the present value of both principal and annuity; that is, we must pay $P + A_0$.

But

$$\begin{aligned} P &= \$1000 \times v^{16} \text{ at } 2\tfrac{1}{2}\% \\ &= \$1000 \times 0.6736249 \\ &= \$673.62; \end{aligned}$$

and

$$\begin{aligned} A_0 &= \$30 \cdot a_{\overline{16}|} \text{ at } 2\tfrac{1}{2}\% \\ &= \$30 \times 13.0550027 \\ &= \$391.65. \end{aligned}$$

Hence the value is \$673.62 + \$391.65, or \$1065.27.

181. General Formula. We may now deduce a general formula for the case considered in § 180.

If F is the face of the bond, n the number of interest periods to elapse, r the interest rate specified in the bond, and i the basis rate or rate of income on the investment, then $\frac{1}{2}Fr$ is the interest received for each interest period.

We have for the present value of the face

$$P = Fv^n = \frac{F}{(1+\frac{1}{2}i)^n}.$$

The present value of the interest payments is

$$A_0 = \frac{Fr}{2} \cdot a_{\overline{n}|} \text{ at } \frac{1}{2}i = \frac{Fr}{i}\left[1 - \frac{1}{(1+\frac{1}{2}i)^n}\right].$$

Then the value of the bond is evidently $P + A_0$.

182. Expenses of Transfer. Since the rate of income to the buyer is to be $\frac{1}{2}i$ per period, the value of the bond as found in §§ 180 and 181 must represent the total investment. Hence any such expenses as brokerage and transfer tax must be deducted from the price paid for the bond.

Exercise 76. Separate Valuation of Principal and Annuity

In Exs. 1–4 find the value or basis price of each bond by the method of §§ 180 and 181:

1. Face of bond, $1000; interest, 6% J. and J.; time to run, 15 yr.; required return on the investment, 5%.

2. Solve Ex. 1 for a return of 7%.

3. Face of bond, $1000; interest, 5% A. and O.; time to run, 18 yr.; required return on the investment, 6%.

4. Face of bond, $500; interest, 4% M. and S.; time to run, 10 yr.; required return on the investment, 4½%.

5. In Ex. 4, if the transfer tax and the brokerage amount to $1.50, how much will be realized by the seller of the bond?

6. Solve Ex. 1 on the condition that the bond runs 5 yr. instead of 15 yr.

7. Solve Ex. 3 on the condition that the bond runs 8 yr. instead of 18 yr.

8. Write a statement of the effect produced on the basis price of a bond by shortening the time to maturity, the bonds being bought above par, and give the reason for this effect. Do the same for bonds bought below par and for bonds bought at par.

9. On Jan. 1, 1921, a man bought a $1000 bond bearing 6% interest J. and J. If the bond yielded him 6.5% return on the investment and is due Jan. 1, 1927, find the price at which he bought it.

183. Valuing Excess or Deficit of Income. We may also find the basis price of a bond by valuing the excess or deficit of income. For example, if I wish to receive 5% on my investment, I can afford to pay \$1000 for a 5% bond; but if the bond bears only 4%, I cannot afford to pay as much. In this latter case I should evidently deduct the present value of an annuity of \$10 per annum withheld. Finding the discount or premium on a bond is thus reduced to finding the present value of an annuity.

184. Illustrative Examples. 1. Find the basis price of a \$1000 bond bearing 6% J. and J., due Jan. 1, 1928, bought Jan. 1, 1920, to yield 5%.

The excess of income each period is \$30 – \$25, or \$5, and so the purchaser must pay a premium equal to the present value of this annuity

Hence
$$\begin{aligned} A_0 &= \$5 \cdot a_{\overline{16|}} \text{ at } 2\tfrac{1}{2}\% \\ &= \$5 \times 13.0550027 \\ &= \$65.28, \text{ the premium.} \end{aligned}$$

Therefore the basis price is \$1000 + \$65.28, or \$1065.28, and the bond will be quoted at 106.53.

Compare this solution with the one given in § 180.

2. Find the basis price of a \$1000 bond bearing 4% A. and O., having $6\frac{1}{2}$ yr. to run, and bought to yield 5.6%.

The deficit of income each period is \$28 – \$20, or \$8, and therefore the purchaser must buy the bond at a discount equal to the present value of this annuity.

Using § 131,
$$A_0 = \frac{R}{i}\left[1 - \frac{1}{(1+i)^n}\right].$$

Here $i = \frac{1}{2}$ of 5.6%, or 2.8%, and we find the discount D as follows:

$$D = \frac{8}{0.028}\left(1 - \frac{1}{1.028^{13}}\right).$$

By logarithms,
$$\frac{1}{1.028^{13}} = 0.698379,$$

and hence
$$D = 86.18.$$

Therefore the bond must sell for \$1000 – \$86.18, or \$913.82.

Exercise 77. Valuing Excess or Deficit of Income

1. Find the basis price of a $1000 bond, bearing 5% J. and J., having 8 yr. to run, and bought to realize 4%.

2. Find the premium and the basis price of a $500 bond, bearing 6% F. and A., having 7 yr. to run, and bought to realize $4\frac{1}{2}$%.

3. Find the discount and the basis price of a $1000 bond, bearing 4% M. and S., having 11 yr. to run, and bought to realize 5%.

4. Find the basis price of a $1000 bond, bearing 5% J. and J., due in 20 yr. but redeemable at par at the option of the debtor in 10 yr. or on any interest day thereafter, and bought to realize 4% on the investment.

When, as in this case, the basis rate is less than the bond rate, the earliest possible date of redemption is used in finding the value.

5. In Ex. 4 find the basis price if the bond is bought to realize 6% instead of 4%.

When, as in this case, the basis rate exceeds the bond rate, the latest possible date of redemption is used in finding the value.

6. In Ex. 4 find the basis price if the bond is redeemable at 105% of its face instead of at par.

Use the method of separate valuation of principal and income (§ 180).

7. Find the price that must be paid for a $100 bond bearing 6% J. and J., due in $5\frac{1}{2}$ yr., bought to realize 5.8%.

8. If the bond in Ex. 7 is bought to realize 5.4%, find the purchase price.

9. Find the price that must be paid for a $100 bond bearing $5\frac{1}{2}$% J. and J., due in $2\frac{1}{2}$ yr., and bought to realize 6.2%, the transfer expenses being $0.75.

10. Find the basis price of a $100 bond bearing 5%, redeemable in 5 yr. at 104% of the face, if bought to realize 6%.

185. Schedule of Amortization or Accumulation. When a bond which is redeemable at par is bought at a premium, a schedule may be constructed showing the amortization of the premium out of the surplus interest received. The sum of the amortization installments must be equal to the premium paid, thus making good the difference between the basis price and the redemption price.

Similarly, if the bond is bought at a discount, the interest received is less than the interest realized. The differences are then regarded as added to the value of the bond, thus increasing this value by accumulation until the bond reaches par at maturity.

186. Illustrative Examples. **1.** Construct the schedule of amortization of the premium on a \$100 bond bearing 6% J. and J., due Jan. 1, 1927, bought Jan. 1, 1924, to yield 5%.

As in § 184, we find that the premium is $\$0.50 \cdot a_{\overline{6|}}$ at $2\frac{1}{2}\%$, or \$2.754. Hence the basis price or book value Jan. 1, 1924, is \$102.754. The schedule is then as follows:

DATE	INTEREST, 6%		INCOME, 5%		AMORTIZATION		BOOK VALUE	
1924, Jan. 1							102	754
July 1	3		2	569		431	102	323
1925, Jan. 1	3		2	558		442	101	881
July 1	3		2	547		453	101	428
1926, Jan. 1	3		2	536		464	100	964
July 1	3		2	524		476	100	488
1927, Jan. 1	3		2	512		488	100	
Totals	18		15	246	2	754		

As a check we see that the total of the amortization column must be equal to the premium, and also that \$15.246, the total net income at 5%, added to \$2.754, the total amortization, must be equal to \$18, the total income at 6%.

2. Construct the schedule of accumulation of the discount on a \$100 bond bearing 6% J. and J., due Jan. 1, 1926, bought Jan. 1, 1923, to yield 7%.

As in § 184, we find the discount by valuing the deficit of income. That is,

$$\begin{aligned} D &= \$0.50 \cdot a_{\overline{6|}} \text{ at } 3\tfrac{1}{2}\% \\ &= \$0.50 \times 5.3285530 = \$2.664, \end{aligned}$$

and hence the basis price or book value on Jan. 1, 1923, is \$97.336.

The schedule is then as follows:

Date	Interest, 6%		Income, 7%		Accumulation		Book Value	
1923, Jan. 1							97	336
July 1	3		3	407		407	97	743
1924, Jan. 1	3		3	421		421	98	164
July 1	3		3	436		436	98	600
1925, Jan. 1	3		3	451		451	99	051
July 1	3		3	467		467	99	518
1926, Jan. 1	3		3	483		483	100	
Totals	18		20	665	2	665		

The student should apply the proper checks.

Exercise 78. Amortization or Accumulation Schedule

1. Construct the schedule of amortization of the premium on a \$1000 bond bearing 5% J. and J., due Jan. 1, 1926, bought Jan. 1, 1922, to yield 4%.

2. Construct the schedule of accumulation of the discount on a \$1000 bond bearing 5% J. and J., due Jan. 1, 1928, bought Jan. 1, 1924, to yield 6%.

3. Construct the schedule of amortization of the premium on a \$1000 bond bearing 5% A. and O., due Oct. 1, 1930, bought Apr. 1, 1925, to yield 4%.

187. Bonds bought between Interest Days. If a bond is bought between interest days, there are two methods in practical use for finding the basis price. One of these methods will now be considered, and others will be discussed in §§ 190 and 191.

We first find the basis price on the preceding interest day. We then find the amortization or the accumulation for the entire interest period. Finally, we subtract the proportional part of the amortization of premium from the basis price of the preceding interest day or add the proportional part of the accumulation of discount to that basis price. We thus obtain the basis price required, or the investment value of the bond. This price is commonly called the *and-interest price.*

To the and-interest price we add the bond interest for the time elapsed since the preceding interest day and we obtain the price to be paid by the purchaser. This price is commonly called the *flat price.*

188. Illustrative Example. Find the and-interest price and the flat price of the bond mentioned in Ex. 1 of § 186, the purchase being made Mar. 1, 1924, instead of Jan. 1, 1924.

Since $\frac{1}{3}$ of the interest period has elapsed, we take $\frac{1}{3}$ of the amortization, \$0.431, which is \$0.144. Subtracting this from the value on Jan. 1, we have \$102.754 − \$0.144, or \$102.610, as the and-interest price.

Since \$1 of interest has been earned by the bond, we have \$102.610 + \$1, or \$103.61, as the flat price.

A more common way of obtaining the same result is as follows:

Value Jan. 1,	\$102.754
Int. on \$102.754, 2 mo., @ 5%,	.856
Flat price,	\$103.61

Since the buyer does not receive his interest at the time of purchase, this method tends to favor the seller.

In finding the amortization or accumulation for an interest period, first find the basis price on the preceding interest day and then proceed to find this item by the method shown in § 186.

189. Amortization Schedule. The amortization schedule for the bond considered in § 188 begins as follows:

Date	Total Income at 6%		Net Income at 5%		Amortization		Book Value	
1924, March 1							102	610
July 1	2	00	1	713		287	102	323

The remaining entries will be the same as on page 172. The item of \$1.713 is the interest on \$102.754 at 5%, for 4 mo., and the item of \$0.287 is $\frac{2}{3}$ of \$0.431.

190. Present-Value Method. Some bond houses find the selling price by a method commonly called the *present-value method.* They find the basis price at the next interest day, add the interest then due, and find the present value of the sum at the basis rate by the method of simple interest.

Using the example in § 189, we have

Value, July 1, 1923,	\$102.323
Adding the interest due,	3.000
Selling price,	105.323

The present value of this sum is $\$105.323 \div 1.01\frac{2}{3}$, or \$103.596. Here the calculation tends to favor the buyer, since the seller is supposed to invest the present value promptly at the basis rate, in order that he may have \$105.32 at the end of the period; but in actual practice this cannot always be done.

191. Theoretical Method. There is still another method, commonly called the *theoretical method.* By this method we find the compound amount of the book value on the last interest day for the time that has elapsed, at the basis rate, using the formula $A = P(1+i)^n$.

In the example of § 188 the book value on Mar. 1 is $\$102.754 \times 1.025^{\frac{1}{3}}$. Solving by logarithms, we have \$103.603 as the book value.

In Exercise 79 use the method of § 187.

Exercise 79. Bonds bought between Dates

1. Find the and-interest price of a \$500 bond, bearing 6% J. and J., redeemable Jan. 1, 1924, if bought Apr. 1, 1920, to yield 5%.

2. Construct the first three entries of the amortization schedule of the bond in Ex. 1.

3. Find the flat price of a \$1000 bond, bearing 5% J. and J., redeemable Jan. 1, 1929, if bought Oct. 1, 1922, to yield 4%.

4. Construct the first three entries of the amortization schedule for the bond of Ex. 3.

5. Find the and-interest price of a \$1000 bond, bearing 5% A. and O., redeemable Oct. 1, 1926, if bought July 1, 1923, to yield 6%.

6. Construct the schedule of accumulation in full for the bond of Ex. 5, using the result of Ex. 5 as the book value July 1, 1923.

For the Oct. 1 entry, credit the total interest received as \$12.50, and calculate the net income as 3 months' interest at 6% on the book value of April 1, 1923. The difference will be the accumulation to be added to the book value of the first entry.

7. Construct the accumulation schedule for a bond of \$100, bearing 5% J. and J., redeemable July 1, 1929, if bought June 1, 1923, to yield 7%.

8. What is the flat price of the bond of Ex. 7?

9. Find the flat price of a \$1000 Liberty Bond bearing 4% June 15 and Dec. 15, due June 15, 1947, bought Jan. 15, 1918, to yield 4.1%.

10. In Ex. 9, construct the accumulation schedule, extending this schedule only to the entries of June 15, 1919.

11. Find the flat price of the bond mentioned in Ex. 9, if bought Feb. 25, 1919, to realize 4.2%.

192. Installment Bond. Since an installment bond is a promise to pay the principal and the interest in equal periodic installments, finding the basis price of such a bond to yield the rate i on the investment is simply a problem in annuities.

Consider the case of an installment bond of face value \$1000, to be paid in ten equal annual installments with interest at 5%, the bond having 6 yr. to run and the purchaser to realize 4% on his investment.

We first find the annual installment, which is simply the annual rent R. Since $A_0 = Ra_{\overline{10}|}$ at 5%, and $A_0 = \$1000$, we have

$$R = \$1000 \cdot \frac{1}{a_{\overline{10}|}} \text{ at } 5\%$$
$$= \$1000 \times 0.1295046 = \$129.505.$$

The purchaser is thus buying an annuity of \$129.505 for 6 yr. on a 4% basis. Again using the formula $A_0 = Ra_{\overline{n}|}$, we have

$$A_0 = \$129.505 \times 5.2421369 = \$678.88.$$

Exercise 80. Installment Bonds

1. The principal of a bond and interest at 6% are to be paid in five equal annual installments. Find the price that should be paid on the day of issue to net the purchaser 4%.

2. The principal of a bond and interest at 6% are to be paid in twelve equal semiannual installments. If the bond still has 5 yr. to run, how much should be paid for it so as to yield to the investor 5% converted semiannually.

3. Find the purchase price of a \$100 bond bearing 5% interest, the principal and interest to be paid in ten equal semiannual installments, the purchaser realizing $4\frac{1}{2}\%$ on his investment and the bond having 4 yr. to run.

4. Find the price that should be paid for the bond mentioned in Ex. 1, if bought the day the first annual installment was paid.

193. Bond Tables. From § 177 we see that the calculable factors affecting the price of a bond are the time it has to run, the rate of interest it pays, and the rate to be realized on the investment. These three elements are usually shown in relation to one another in *bond tables.*

Such tables are calculated on a semiannual basis, a separate page being allowed to every half year to 50 yr. In the case of bonds running for a longer time, a separate page is allowed to every 5 yr. and the tables extend to 100 yr. There is, of course, a variation in this arrangement in the case of different tables.

The following is a portion of a page of a bond table:

15 YEARS. INTEREST SEMIANNUALLY

Per Cent Yield per Annum	Bond Rate						
	3%	$3\frac{1}{2}$%	4%	$4\frac{1}{2}$%	5%	6%	7%
4.10	87.77	93.33	98.89	104.45	110.01	121.13	132.25
$4\frac{1}{8}$	87.51	93.06	98.61	104.16	109.71	120.82	131.92
4.20	86.75	92.27	97.79	103.31	108.84	119.88	130.93
$4\frac{1}{4}$	86.24	91.74	97.25	102.75	108.26	119.26	130.27
4.30	85.74	91.22	96.71	102.19	107.68	118.65	129.62
$4\frac{3}{8}$	84.99	90.45	95.91	101.36	106.82	117.74	128.65
4.40	84.75	90.19	95.64	101.09	106.54	117.43	128.33
$4\frac{1}{2}$	83.77	89.18	94.59	100.00	105.41	116.23	127.06
4.60	82.80	88.18	93.55	98.93	104.30	115.05	125.80
$4\frac{5}{8}$	82.56	87.93	93.29	98.66	104.02	114.76	125.29
4.70	81.85	87.19	92.53	97.86	103.20	113.88	124.56
$4\frac{3}{4}$	81.38	86.70	92.02	97.34	102.66	113.30	123.94
4.80	80.91	86.21	91.52	96.82	102.12	112.73	123.33
$4\frac{7}{8}$	80.21	85.49	90.77	96.04	101.32	111.87	122.42
4.90	79.98	85.25	90.52	95.79	101.05	111.59	122.12

194. Use of Bond Tables. To understand the use of bond tables, suppose that we wish to find the basis price of a bond bearing 5% and having 15 yr. to run, the rate of income on the investment to be $4\frac{1}{4}\%$. We turn to the 15-year page (§ 193) of the bond table and find the 5% column. We then find the line indicated by $4\frac{1}{4}\%$ at the left, and we read the price, 108.26. Hence the price of a \$100 bond is \$108.26, and the price of a \$1000 bond is \$1082.60.

If we wish to know the yield on a 5% bond having 15 yr. to run, the basis price being 102.66, we follow the 5% column until we reach 102.66. Now looking in the left-hand column we find $4\frac{3}{4}$. Hence the yield is $4\frac{3}{4}\%$.

Sometimes it is necessary to interpolate, as in a table of logarithms, the result being then an approximation.

For example, if we wish to know the rate of yield on a 15-year bond bearing 5% to be bought at 103.80, we look in the 5% column and find the prices 104.02, yielding $4\frac{5}{8}\%$, and 103.20, yielding 4.7%. Interpolating as in logarithms, we find that 103.80 yields 4.645%.

A more accurate but less convenient method is to use logarithms.

Exercise 81. Bond Tables

1. From the table determine the price that should be paid for a bond bearing 6% J. and J., redeemable in 15 yr., to yield 4.8% on the investment.

2. If $4\frac{1}{2}\%$ bonds due in 15 yr. are selling at 97.86, what rate do they yield the investor?

3. An investor is willing to buy a 15-year bond bearing 6% if he can realize $4\frac{3}{4}\%$ on his investment. What price should he pay?

4. Using logarithms, apply the method of § 183 to finding the price that should be paid for a bond having 15 yr. to run, the bond bearing $4\frac{1}{2}\%$ J. and J. and the yield being 4.8%; and compare the result with that given in the bond table.

195. Bond Valuation by Logarithms. When the amount involved is large, it often happens that interpolation in a bond table is not accurate to a degree sufficient to give the basis price or to find the yield rate. If the item desired falls between the items of the table, a result more nearly accurate can be obtained by the use of logarithms.

Find the price that I can afford to pay for a $1000 bond bearing 6% J. and J., having 12 yr. to run, to realize 4.55%.

Income at bond rate	$30.00
Income at basis rate	22.75
Excess income	$7.25

The premium is the present value of an annuity of $7.25 a period, for 24 periods at 0.02275 per period.

Hence we have $$A_0 = \frac{\$7.25}{0.02275}\left(1 - \frac{1}{1.02275^{24}}\right).$$

Using logarithms, we have $A_0 = \$132.953$.
Hence the bond costs $1132.95, or is bought at 113.295.

Exercise 82. Valuation by Logarithms

1. Find the basis price of a $1000 bond, having $15\frac{1}{2}$ yr. to run, bearing 5% J. and J., and bought to realize 5.4%.

2. Find the basis price of a $100 bond, having 16 yr. to run, bearing 4% J. and J., and bought to realize 5.2%.

3. Find the basis price of a $1000 bond, having $9\frac{1}{2}$ yr. to run, bearing 5% J. and J., and bought to realize 4.75%.

4. Bonds bearing 6%, payable semiannually and having 15 yr. to run, are quoted at 107.5. A bond table shows that the yield rate is between 5.25% and 5.3%. Interpolation gives the yield rate as 5.279%. Test this by logarithms and see if the result is 107.5 for basis price. Test also several values near 5.279% to arrive at a more nearly correct yield rate.

Exercise 83. Review of Chapter X

1. Using logarithms, calculate the bond-table item shown on page 178 for a $4\frac{1}{2}\%$ column, to yield 5%.

2. Find the basis price of a $100 bond having 4 yr. to run, bearing 5% J. and J., bought Jan. 1 to yield 6%.

3. Construct the schedule of accumulation for the bond in Ex. 2, the bond being bought Jan. 1 of this year.

4. Find the value of a $1000 bond bearing 6% J. and J., due Jan. 1, 1932, if bought Oct. 1, 1921, to yield 5%.

5. Find the value of the bond in Ex. 4, if yielding 5.4%.

6. Find the entries for Oct. 1, 1921, and Jan. 1, 1922, of the amortization schedule of the premium on the bond in Ex. 4.

7. The Central Realty Corporation issued a series of $500 bonds bearing 6%, interest J. and D., principal due Dec. 1, 1933. Find the value, on a 5% basis, June 1, 1920.

8. Find the and-interest price of the bond in Ex. 7, on a 5% basis, Oct. 1, 1920.

9. Calculate the price quotation in the bond table on page 178 for the intersection of the column of the 3% bond rate and the line of the 5% yield.

10. By using logarithms calculate the price quotation on the $7\frac{1}{2}$-year page of a bond table for the intersection of the column of the 5% bond rate and the line of the $4\frac{7}{8}\%$ yield.

11. In Ex. 10, find the quotation at the intersection of the 5% column and the $4\frac{3}{4}\%$ line.

12. Find the quotation on the 50-year page of a bond table at the intersection of the column of the 5% bond rate and the line of the $4\frac{5}{8}\%$ yield; also find the quotation for the 6% column and the $4\frac{5}{8}\%$ line.

Exercise 84. Review of Chapters I–X

1. Given that $Vr^n = V'$, solve for $\log r$.

2. Given that $V = \$4000$, $V' = \$300$, and $n = 15$, find the value of r, using the formula for $\log r$ found in Ex. 1.

3. Show that Ex. 1 is the problem of finding r in a geometric progression in which l, a, and n are given.

4. The equation of Ex. 1 is essentially that of the reducing-balance method of reckoning depreciation. What does r represent if used for this purpose? From the formula given in the discussion of that method derive a formula for n.

5. Using the result of Ex. 4, find the time in which a machine valued at \$5000 will reduce to a scrap value of \$250, if 20% of the remaining value is written off each year.

6. At what rate will money at compound interest increase 50% of itself in 8 yr.? double itself in 15 yr.?

7. A corporation borrowed \$2,500,000, at $4\frac{1}{2}$%, on 20-year bonds. To meet the principal when due it set aside a sinking fund which accumulated at 4%. Find the total annual charge for carrying the bond issue, including interest and sinking fund.

8. In Ex. 7 construct the amortization schedule for the first 5 yr., showing the accumulation of the sinking fund.

9. A city, needing \$100,000 to finance the building of a city hall, plans to issue serial bonds bearing 5% and to arrange that the annual burdens for interest and for bonds redeemed shall be approximately equal. The last bond is to be redeemed 15 yr. from the date of issue. If the denominations of the bonds are \$1000, \$100, and \$50, find the amount of bonds to be redeemed at the end of each of the first 5 yr.

10. If \$1,000,000 were bequeathed in trust to a city to be invested at compound interest at 4% until the amount were \$500,000,000, in how many years would this sum be realized?

CHAPTER XI

LIFE INSURANCE

196. Insurance. The business of assuming the risk of financial loss by death, bodily injury, fire, tornado, flood, or other contingent event is called *insurance.*

The risk of insurance is assumed by an insurance company in return for a consideration paid by the person insured.

197. Policy. A contract of insurance stating the terms under which the insurer agrees to pay a certain sum to the insured or to the beneficiary named in the contract is called a *policy.*

198. Premium. The sum paid by the insured to the insurance company for assuming the risk stated in the policy is called the *premium.*

If the premium is all paid at one time, it is called a *single premium*; but if it is paid in annual installments, each installment is called an *annual premium.*

For example, if a man takes out a policy with a life-insurance company and agrees to pay $17.52 a year for life, in return for which the company agrees to pay $1000 to his wife or children upon his death, he has taken out an ordinary life policy and his annual premium is $17.52.

No company would be justified in issuing such a policy to only one person; but if a large number of such contracts are made, the deaths will occur with sufficient regularity to enable a company to issue such policies with entire safety. The fundamental principle of insurance is that of probability based on a large number of cases.

The annual premium that must be paid for a life policy increases with the age of the insured at the time he takes out his policy.

199. Life Insurance. In life insurance the benefit to be paid at the death of the insured to his estate, or to the beneficiary named in the policy, may be provided by the agreement of all the persons insured in that company to contribute a sum each year sufficient to pay the death losses of that year, or by the agreement of each person insured to pay a specified premium based upon experience.

The first of these plans is known as *assessment insurance* or the *cash-disbursement plan.*

The second plan, known as the *level-premium plan,* depends on the fact that in a sufficiently large and scattered group the number of deaths in any one year can be foretold to a high degree of accuracy. Statistics relating to a small group of individuals in one city would be useless for such a purpose, but if gathered from 100,000 cases scattered about the country and of various ages, these statistics would be very reliable.

200. Basis of Life Insurance. We may suppose that 200 men know that 20 of them will die the first year, 40 the second year, 60 the third year, and the remaining 80 the fourth year. If they agree to contribute such a sum at the beginning of each year that $1000 can be paid to the family of each deceased, we shall have, neglecting interest, the following:

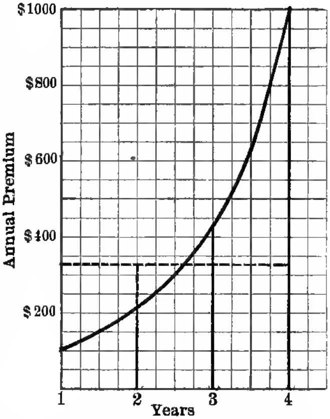
YEAR	CONTRIBUTORS	DEATHS	DEATH BENEFITS
1	200	20	$20,000
2	180	40	40,000
3	140	60	60,000
4	80	80	80,000
Totals	600	200	$200,000

To share the burden equally, each living man should pay at the beginning of each year $200,000 ÷ 600, or 333.33\frac{1}{3}$. The first year the total would be $66,666.66$\frac{2}{3}$, leaving $46,666.66$\frac{2}{3}$ for future benefits.

201. Assessment Plan. The assessment plan makes the contributions light during the first years of a company's existence, but they become heavier as the company becomes older. This condition would tend to be removed after a company had been long in existence.

This condition is seen by considering again the case discussed in § 200. If there is contributed annually just enough to meet the death benefits of the year, the individual contributions in the first year are $100, while in the fourth year they are $1000, as is seen in the following schedule:

Year	Contributors	Deaths	Each Pays	
1	200	20	100	
2	180	40	222	22
3	140	60	428	57
4	80	80	1000	



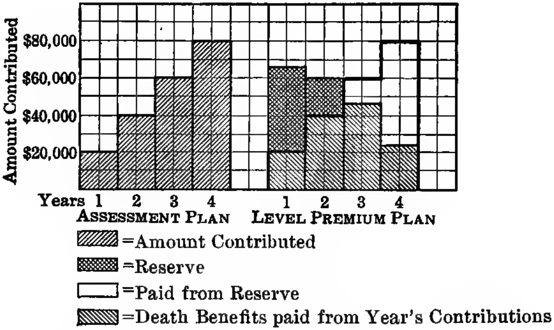
A comparison of the two systems is made clearer by means of the accompanying graphs of level premiums and assessment premiums.

Since interest is not considered in this elementary sketch, it is evident that the total contributions in the two plans must be equal. The distribution of the burden under the level-premium plan is shown approximately by the dotted line, while that under the assessment plan is shown by the curve.

The increasing premiums under the assessment plan have caused the failure of very many attempts at this form of life insurance.

If the assessments were paid at the end instead of the beginning of the year, they would be larger than those stated above.

202. Comparison of Total Contributions. If we neglect the question of interest, the total contributions paid by the insured parties is the same under the assessment plan as under the level-premium plan. The distribution of each total referred to in § 201 is shown in the following graph:



In each figure the shaded area represents the total contribution made to the company each year.

In the level-premium plan it is evident that the major part of the fund is contributed while the majority of the insured are living. It will also be seen that the greatest total yearly contribution is made by the entire 200 men in the first year, and the least by the surviving 80 men in the fourth year.

By counting the squares it may be seen that the shaded area in the second graph, designated as "Reserve," is equal to the unshaded area, designated as "Paid from Reserve." The essential feature of the old-line, level-premium, legal-reserve companies is that these companies provide a reserve during the first years, when the death benefits of the insured group are light, to assist in meeting the heavier death benefits of the later years.

In practice the excess contributions of the first years, constituting the reserve, draw interest. The compound-interest accumulations thus permit of a smaller premium than would be necessary if interest were not considered.

203. Contributions of the Insured. From the problem considered in §§ 201 and 202 it is evident that if the funds paid in at the beginning of the first year were put at interest we should have more than \$66,666.66$\frac{2}{3}$ by the end of the year; that is, less than \$333.33$\frac{1}{3}$ would be contributed by each of the insured, since less than \$66,666.66$\frac{2}{3}$ would accumulate enough to pay the \$20,000 of death dues and provide a reserve, together with the interest and the subsequent contributions, to make up the losses in the following years. If 3% can be realized, the present value of \$66,666.66$\frac{2}{3}$ on a 3% basis would suffice for the death benefits of the first year and for a reserve for future years. For this reason the level premium could be reduced. The method of finding this premium will be considered in § 211.

204. Mortality Table. If a large number of individuals are observed and a record is kept of the number who die at each age, a surprising degree of conformity to the observed death rate is found in the case of any other large group of individuals of the same general character living under the same conditions. The theory of probability, considered in advanced algebra, indicates the limits within which the actual death rate will conform to the expected death rate calculated from tables based on extended observation.

An experiment in probabilities can be easily made as follows: Draw on a piece of paper 25 parallel lines, $\frac{1}{4}$ in. apart, making the middle line heavier than the rest. Then try to drop a pencil, point downwards, from a position 3 ft. above the paper, so as to hit the middle line. The dots made will be scattered about equally on both sides, the fewer dots being at the greater distance, and those near the corresponding lines becoming the more nearly equal as the number of trials becomes larger.

The law of action of the forces determining the fall of the pencil is shown when many trials are made and the results observed, and the same is true with respect to death occurrences. If the groups of individuals are large, the death rate is determined with close approximation to accuracy.

205. American Experience Table. The American Experience Table is the work of the actuarial department of one of the largest life-insurance companies in this country. It represents approximately the experience of the company with respect to insured lives which were selected after a careful medical and personal examination. It traces the mortality of 100,000 persons from the age of 10 yr. to the age of 95 yr., when the last one died, and it is the basis of calculation of premiums prescribed by many states of the United States. An experience table, differing somewhat from this table, is used in England.

AMERICAN EXPERIENCE TABLE OF MORTALITY

At Age x	Number Surviving l_x	Deaths During Year d_x	At Age x	Number Surviving l_x	Deaths During Year d_x	At Age x	Number Surviving l_x	Deaths During Year d_x
10	100,000	749	25	89,032	718	40	78,106	765
11	99,251	746	26	88,314	718	41	77,341	774
12	98,505	743	27	87,596	718	42	76,567	785
13	97,762	740	28	86,878	718	43	75,782	797
14	97,022	737	29	86,160	719	44	74,985	812
15	96,285	735	30	85,441	720	45	74,173	828
16	95,550	732	31	84,721	721	46	73,345	848
17	94,818	729	32	84,000	723	47	72,497	870
18	94,089	727	33	83,277	726	48	71,627	896
19	93,362	725	34	82,551	729	49	70,731	927
20	92,637	723	35	81,822	732	50	69,804	962
21	91,914	722	36	81,090	737	51	68,842	1001
22	91,192	721	37	80,353	742	52	67,841	1044
23	90,471	720	38	79,611	749	53	66,797	1091
24	89,751	719	39	78,862	756	54	65,706	1143

At Age x	Number Surviving l_x	Deaths During Year d_x	At Age x	Number Surviving l_x	Deaths During Year d_x	At Age x	Number Surviving l_x	Deaths During Year d_x
55	64,563	1199	70	38,569	2391	85	5485	1292
56	63,364	1260	71	36,178	2448	86	4193	1114
57	62,104	1325	72	33,730	2487	87	3079	933
58	60,779	1394	73	31,243	2505	88	2146	744
59	59,385	1468	74	28,738	2501	89	1402	555
60	57,917	1546	75	26,237	2476	90	847	385
61	56,371	1628	76	23,761	2431	91	462	246
62	54,743	1713	77	21,330	2369	92	216	137
63	53,030	1800	78	18,961	2291	93	79	58
64	51,230	1889	79	16,670	2196	94	21	18
65	49,341	1980	80	14,474	2091	95	3	3
66	47,361	2070	81	12,383	1964			
67	45,291	2158	82	10,419	1816			
68	43,133	2243	83	8,603	1648			
69	40,890	2321	84	6,955	1470			

Actuaries use tables that give the columns here shown, together with additional columns known as *commutation columns*, by means of which the calculation of the premium on a life policy, the value of an *endowment of 1 per annum*, and similar items are rendered easier than by the method shown in § 211. This theory is too difficult to be treated in a book of this kind. In the following pages only the general principles involved in calculating a premium are discussed.

The table used in England, although computed independently of the American Experience Table, shows a remarkable degree of agreement with the latter with respect to the number of deaths in each year. Long experience has shown that each of these tables is very reliable.

206. Term Insurance. Life insurance covering a definite number of years is called *term insurance.* The person is insured only to the end of the time specified.

If, for example, a man takes out term insurance at age 25 for 10 yr., it means that if he dies during this time, his beneficiary will receive the face of the policy, but if he lives to age 35, the contract then lapses. He has bought insurance for 10 yr. at a small annual cost.

207. Ordinary Life Policy. A life-insurance policy that remains in force, if the premiums are paid, until the death of the insured, is called an *ordinary life policy.* When the insured dies, his beneficiary receives the stipulated benefit.

208. Limited-Payment Life Policy. A life-insurance policy that provides for the payment of a limited number of annual premiums, the protection extending through the life of the insured, is called a *limited-payment life policy.*

If a man takes out a straight life policy, he pays continuously during life, and at death his estate receives the face of the policy. If he has a 20-payment life policy, he pays a larger premium than on a straight life policy, but stops paying after 20 premiums have been paid. The compound-interest accumulations and the excess of his larger premiums will take care of the charges for the following years.

209. Endowment Policy. A life-insurance policy of term insurance, combined with a policy giving the right to receive the face if the insured survives the term, is called an *endowment policy.* The calculation of the premium depends on the nature of the contract or policy.

A pure endowment contract would call for certain payments during the contract period in return for the right to receive the stipulated sum, say $1000, at the end of the time if the insured survives. If the insured were to die, his estate would receive nothing. The so-called endowment policies provide that if the insured survives, he receives the $1000; but if he dies before the end of the period, his beneficiary receives the $1000 at his death. There are really the two contracts, one of pure endowment and the other of term insurance, the contract time being the same.

210. Calculating a Premium. The work of calculating premium tables is done by specially trained actuaries. These actuaries use the theory of probabilities and various other features of higher mathematics that cannot be explained in a book of this nature. While the method in § 211 is longer than that used by actuaries, it serves to make the subject clear.

211. Illustrative Example. Without loading for expenses, calculate the level premium on a $1000 term policy for 10 yr., at age 35, on a 3% basis.

In the following schedule, columns 1, 2, and 3 are from the American Experience Table. It is assumed that we start with 81,822 persons, this being the number given at age 35 in the table. Since the policy is for $1000, column 4 shows the amount (C) in death claims paid each year. Column 6 shows the present value of these death claims, and uses the formula $V = Cv^n$. Column 7 shows the present value of a contribution of $1 from each of the survivors of each year. Columns 6 and 7 are accurate to the nearest cent.

1	2	3	4	5	6	7
AGE	SURVIVORS AT AGE x	NUMBER DYING	DEATH CLAIMS	YEARS IN FORCE	PRESENT VALUE OF DEATH CLAIMS	PRESENT VALUE $1 CONTRIBUTIONS
x	l_x	d_x	C	n	$V = Cv^n$	$V' = l_x v^n$
35	81,822	732	732,000	1	710,679.62	81,822.00
36	81,090	737	737,000	2	694,693.18	78,728.16
37	80,353	742	742,000	3	679,035.14	75,740.41
38	79,611	749	749,000	4	665,476.76	72,855.35
39	78,862	756	756,000	5	652,132.25	70,067.86
40	78,106	765	765,000	6	640,675.49	67,374.92
41	77,341	774	774,000	7	629,332.82	64,771.87
42	76,567	785	785,000	8	619,686.22	62,255.98
43	75,782	797	797,000	9	610,834.11	59,823.01
44	74,985	812	812,000	10	604,204.25	57,469.76
					6,506,749.84	690,909.32

212. Value of the Premium. If to each person enumerated in column 2 under § 211 the company gives the right to receive \$1000 at death, the present value of all these rights being \$6,506,749.84, as shown in the total of column 6, and if the company in exchange has the right to an annual premium of p dollars per annum from each of the insured, the present values of these two must balance.

The present value of \$1 per annum from each of the insured is \$690,909.32, as shown in the total of column 7, and so the present value of p dollars per annum from each of the insured is $690{,}909.32p$.

Therefore $690{,}909.32p = \$6{,}506{,}749.84$

and $p = \$9.42.$

The annual premium is \$9.42 without any loading; that is, without any addition for expenses of administration. The rate quoted by one of the large companies for the above policy is \$10.30, the addition of 88¢ in this case being known as *loading the premium.*

213. Whole-Life Premium. If we use the method of § 212 to calculate the premium on an ordinary or whole-life policy, the seven columns of § 211 would be continued to the end of the table, at age 95. The quotient of the total of column 6 divided by that of column 7 gives the annual premium before loading.

214. Endowment-Policy Premium. In finding the premium on an endowment policy the columns for a term policy for the period of the endowment are first calculated as in § 211. To the total of column 6 there is added the present value of \$1000 paid to each survivor at the end of the time, and the sum is then divided by the total of column 7. It is evident that the longer the period covered by the endowment the more will the compound-interest accumulations reduce the cost of the pure endowment feature.

215. Illustrative Example. Find the annual premium on a 10-year endowment policy for $1000, at age 35, on a 3% basis.

As in § 211, the present value of all the death claims that will occur is $6,506,749.84, and by the table there will be 74,173 survivors, each to receive $1000. The present value of 74,173 × $1000, or $74,173,000, due in 10 yr., is $55,191,676.84. Hence the total present value of the obligation assumed by the company is $6,506,749.84 + $55,191,676.84, or $61,698,426.68.

If this amount is balanced by the annual premiums of p dollars from each of the survivors, we have

$$690{,}909.32\,p = \$61{,}698{,}426.68,$$

and $p = \$89.30$, the premium without loading.

216. Terminal Reserve. The amount accumulated by an insurance company as a reserve to meet future death claims of the insured group, and remaining on hand each year after paying the death benefits of that year, is called the *terminal reserve* of the year.

Taking the data in § 215, the method of calculation is as follows:

1	2	3	4	5
AGE	SURVIVORS AT AGE x	PREMIUMS FROM SURVIVORS	AMOUNT AT 3%	TERMINAL RESERVE
x	l_x			
35	81,822	$7,306,704.60	$7,525,905.74	$6,793,905.74
36	81,090	7,241,337.00	14,456,300.02	13,719,300.02
37	80,353	7,175,522.90	21,521,667.61	20,779,667.61
38	79,611	7,109,262.30	28,725,597.81	27,976,597.81
39	78,862	7,042,376.60	36,069,543.64	35,313,543.64

Column 2 gives the number of survivors at the beginning of the year; column 3, the premiums from these survivors; column 4, the amount at 3% of the year's premium plus the terminal reserve of the preceding year; column 5 is the same as column 4 less all death claims. The total death claims are taken from the table on page 191.

Exercise 85. Calculating Annual Premiums

1. Calculate the annual premium without loading, on a 3% basis, of a straight life policy for $1000 at age 85, using the American Experience Table.

Of course the group of survivors at this age, shown in the table, is too small for the law of averages to work. The example will give the necessary insight into the method, however. Insurance companies do not insure persons at age 85.

2. In § 216 carry each of the five columns to age 45.

3. From the results of Ex. 2 find the reserve value of a 10-year endowment policy taken out at age 35, at the end of the sixth year.

4. Using the table of § 211, column 6 shows that the death benefits to be met in the 10 yr. have a present value of $6,506,749.84. If one of the 81,822 persons insured at age 35 desired to pay a single premium at once to cover his future benefit, how much would it be?

5. Calculate the annual premium on a 5-year term policy for $1000, on a 3% basis, at age 25, using the American Experience Table.

6. If a man takes out a 5-year term policy at the cost per annum calculated in Ex. 5 and dies at the end of the third year, how much more cash will his estate receive than could have been accumulated in a savings bank that paid 4% compounded semiannually?

7. Calculate the total amount that the company in Ex. 5 can accumulate from the annual premiums of a policyholder in 5 yr. if it can invest them at 4% compounded annually.

8. If an annual premium of $35.26 is invested at 4% compounded annually, how much will it amount to at the end of 20 yr.?

217. Insurance Rates. As before indicated, the rate per annum on a life-insurance policy depends on the age of the insured and on the kind of contract. The following table shows the rates per $1000 charged by a prominent old-line legal-reserve company at different ages from 20 yr. to 39 yr.

Age	Annual Premium on $1000							
	Life	Life 10 Payments	Life 15 Payments	Life 18 Payments	Life 20 Payments	Endowment 10 Years	Endowment 15 Years	Endowment 20 Years
20	$17.52	$41.38	$30.94	$27.53	$25.85	$101.27	$64.63	$46.76
21	17.90	42.06	31.45	27.99	26.29	101.35	64.71	46.85
22	18.30	42.74	31.98	28.46	26.72	101.42	64.80	46.95
23	18.73	43.46	32.52	28.96	27.18	101.51	64.89	47.06
24	19.18	44.20	33.08	29.46	27.67	101.59	64.99	47.16
25	19.63	44.97	33.68	29.98	28.18	101.69	65.09	47.28
26	20.12	45.77	34.28	30.53	28.69	101.78	65.20	47.40
27	20.64	46.60	34.91	31.11	29.23	101.89	65.32	47.54
28	21.18	47.45	35.57	31.70	29.80	102.01	65.45	47.68
29	21.75	48.35	36.26	32.32	30.39	102.13	65.59	47.84
30	22.35	49.28	36.98	32.96	31.00	102.26	65.74	48.02
31	22.98	50.24	37.71	33.63	31.64	102.39	65.89	48.20
32	23.65	51.25	38.48	34.33	32.31	102.55	66.06	48.41
33	24.35	52.28	39.29	35.07	33.00	102.70	66.24	48.61
34	25.09	53.36	40.13	35.83	33.73	102.88	66.45	48.87
35	25.88	54.49	41.00	36.62	34.50	103.06	66.67	49.13
36	26.72	55.64	41.89	37.46	35.29	103.26	66.89	49.40
37	27.59	56.85	42.84	38.33	36.12	103.47	67.15	49.72
38	28.52	58.11	43.84	39.23	37.00	103.71	67.44	50.07
39	29.51	59.42	44.87	40.18	37.92	103.97	67.75	50.46

The types of policy covered by the table are the life, limited payment, and endowment types. Such tables are given in the rate books issued by insurance companies.

Exercise 86. Cost of Life Insurance

1. Using the table given in § 215 find the annual cost of a 20-year endowment policy for $3000 taken out at age 25.

2. If a man aged 24 takes out a policy of $2000 on his life and lives to age 54, thus making 30 payments, will he contribute more money to the company on the straight life plan or on the 20-payment life plan? How much more?

3. A man takes out a 20-year endowment policy at age 23 and lives to realize the $1000 endowment. Not considering interest, how much does he realize above what he has paid to the company as premiums?

4. A man bought a house, giving a mortgage for $3000 by which he agreed to pay the principal with interest at 6% in 10 equal annual installments. To protect his family during this term he took out a 10-year term-insurance policy for $3000 at $12.28 per $1000. Find the annual charge for discharging the mortgage and carrying the insurance.

5. Two partners, A and B, are engaged in a business that nets them an average yearly profit of $12,000, of which A receives $\frac{2}{3}$ and B receives $\frac{1}{3}$. A is 35 yr. old and B is 36. They agree that each shall take out a straight life policy, naming the other as the beneficiary, in a sum equal to half the average annual net profit of the beneficiary. The premiums are to be paid from the net profits before a division is made between the partners. By how much will the net profits be reduced yearly?

6. If the partners in Ex. 5 had taken out 20-payment life policies instead of straight life policies, how much more would each policy have cost per year?

7. What would have been the added cost in Ex. 5 if 20-year endowment policies had been taken out?

218. Insurance as an Investment. The rate of interest received on the premiums invested in an endowment policy may be calculated by the theory of annuities, as will be seen in the solution of the following example:

A certain company charges \$47.28 per \$1000 per annum for a 20-year endowment policy taken out at age 25, and \$15.10 per \$1000 per annum for a 20-year term policy. Find the rate realized on the former.

Since \$47.28 – \$15.10 represents the cost of the pure endowment, the purchaser is investing \$32.18 per annum in the endowment feature.

These payments constitute an annuity due of 20 terms, the sum or final value being \$1000.

Since \$32.18 invested per annum amounts to \$1000, \$1 invested per annum amounts to \$1000 ÷ 32.18, or \$31.0752, which is therefore $\mathbf{s}_{\overline{20}|}$.

Now, § 129,

$$\mathbf{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1 = \frac{(1+i)^{n+1} - (1+i)}{i}.$$

Looking in the table of $s_{\overline{n}|}$, at the 21-year line, and subtracting 1, we find that $\mathbf{s}_{\overline{20}|}$ lies between 30.9692017 and 32.7831368, which are the values of $\mathbf{s}_{\overline{\ }|}$ at 4% and $4\frac{1}{2}$% respectively. Hence $i = 0.04 + h$, where $h < 0.005$. Substituting in the above equation,

$$31.0752 = \frac{(1+i)^{21} - (1+i)}{i};$$

whence

$$(1+i)^{21} - 32.0752\,i - 1 = 0.$$

Letting $i = 0.04 + h$, we have

$$(1.04 + h)^{21} - 32.0752(0.04 + h) - 1 = 0.$$

Expanding,

$$1.04^{21} + 21 \times 1.04^{20}\,h + \cdots + h^{21} - 2.283008 - 32.0752\,h = 0.$$

Since $h < 0.005$, h^2, h^3, $\cdots$ are so small that we may neglect them in a calculation like this.

Solving the equation

$$1.04^{21} + 21 \times 1.04^{20}\,h - 2.283008 - 32.0752\,h = 0$$

we have

$$h = 0.0003 +.$$

Hence

$$i = 0.0403 \text{ approximately.}$$

Exercise 87. Investment Yield on Policies

1. A man took out a 20-year endowment policy for $1000 at age 23, the annual premium being $47.06, and lived to realize the endowment. If the same company charges $15.20 per annum for a 20-year term policy at age 23, what rate did the man realize on his investment?

2. In a certain company a 15-year endowment policy for $1000 at age 30 costs $65.74 per annum and a 15-year term policy at the same age costs $12.90 per annum. What rate per annum on his premiums, regarded as investments, will be realized by one who buys the endowment policy and lives to the end of the 15 yr.?

3. Mr. B took out a 20-year endowment policy for $1000 at age 23, paying $47.06 per annum. If he had deposited the premiums in a bank at 4% compounded annually, how much more than $1000 would have been to his credit at the end of the 20 yr.? Why should he choose this kind of policy?

4. If money can be kept continuously invested, how much more than $1000 will be the final value of an annuity due of 10 terms, each $92.81 per annum, on a 5% basis? How much better is it to invest the money at 5% than to take out 10-year endowment insurance at age 30, assuming that the 10-year protection costs $9.45 per annum, and the balance of the premium of $102.26 represents the investment?

5. In Ex. 4, how much would have been the advantage to the investor if the $92.81 per annum had been invested at 6% compounded annually?

6. By comparing the premiums for 20-year term insurance and 20-year endowment insurance, a man found that the endowment feature cost $42.72 per annum per $1000. Find the rate of interest realized.

Exercise 88. Review of Chapter XI

1. Using the American Experience Table, calculate on a 3% basis the net level premium that would be charged for a term-insurance policy for $1000 for 3 yr. at age 30.

2. In Ex. 1 find the premium on a 4-year term policy.

3. Using a 3% basis, find the net level premium for a whole-life or straight-life policy at age 93 on the assumption that the risks of death involved are as given in the table and that a company could be found to accept such an unusual risk.

4. If the plan in Ex. 3 were that of assessment insurance, what would the insured have to pay each year, assuming that the assessments could be ascertained and paid in advance, money being worth 3%, and that the actual experience would conform to the American Experience Table?

5. Mr. A took out at age 35 a 20-payment life policy at the rate given in the table in § 217 and lived to age 80. If the company accumulates its money at 4% compound interest, how much will it have on hand to cover the death benefit?

6. In Ex. 5 how much could the company accumulate from Mr. A's premiums if it realized 5% per annum?

7. If a company at the rates given in § 217 realizes 4% compounded annually, how much will it have on hand, at the end of 20 yr., from the premiums and the interest accumulations of a 15-payment life policy and a 20-payment life policy respectively, both being based on age 20 and the insured in each case living to beyond the age of 40 yr.?

8. Mr. A took out an ordinary life policy at age 34, paying the premium given in § 217. He lived to be 86 yr. old. If he had deposited the annual premiums in a savings bank paying 4 % semiannually, how much would be available for his beneficiary at the time of the death of Mr. A?

Exercise 89. Review of Chapters I–XI

1. Find the value of $1.02^3 \times 1.03^2$.

2. Find by logarithms the value of $\dfrac{3.56 \times 2485}{67 \times \sqrt[5]{372}}$.

3. A purchaser's contract for the payment of the balance of $420 on a piano bought on the installment plan calls for the payment of $10 a month together with interest on the unpaid balance at 6% per annum. By summing a certain arithmetic series find the sum of the interest payments.

4. A grocer receives a bill for $1450, terms 60 da. net, 5%, 10 da. Find the highest interest rate he can afford to pay, borrowing the money to pay the bill in 10 da.

5. What is the difference between the final value or sum of an annuity due of $20 a year for 10 yr. and of an ordinary annuity of $15 a year for 13 yr., money being worth 4%?

6. By the reducing-balance method find the annual depreciation charge on an engine costing $1540 when new, and having a life of 15 yr., the scrap value being $80.

7. Find by logarithms the basis price of a $1000 bond having $6\frac{1}{2}$ yr. to run and bearing 6% interest, the bond being bought on an interest day at a price to yield $5\frac{1}{4}$%.

8. Find by logarithms the compound amount of $9500 for $3\frac{1}{2}$ yr. at 4.16%.

9. At what rate per annum must $3500 be compounded to amount to $5000 in 14 yr.?

10. Find the annual contribution to a sinking fund which realizes 4% annually, to amortize a debt of $20,000 in 15 yr.

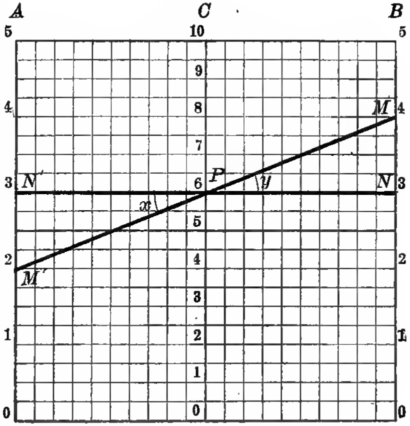
11. How much smaller may the contribution to the sinking fund of Ex. 10 become, if the rate realized is $4\frac{1}{2}$% instead of 4%?

CHAPTER XII

ALIGNMENT CHARTS

219. Alignment Chart. When a series of calculations is to be made, the same process being repeated with different numbers, charts are often constructed such that by means of straight lines results can be obtained mechanically. Such charts are called *alignment charts.*

220. Addition Chart. To illustrate the simplest case, suppose that we wish to construct an alignment chart for adding numbers from 1 to 5. If we draw three equal parallel axes, *A*, *B*, *C*, placing *C* equidistant from *A* and *B*, and having all these axes perpendicular to the line 000, we have our addition chart ready to mark to scale. If we mark off 5 equal spaces on *A* and also on *B*, numbering them from 1 to 5, the straight lines connecting the same numbers on the outer axes will cross *C* at the points giving the doubles of the numbers connected. Also the line joining 2 on *A* and 4 on *B*, or 3 on *A* and 3 on *B*, will cross *C* at point 6, that is, at 2 + 4 or 3 + 3, and similarly for other numbers.



221. Formula $au + bv = c$. In the formula $au + bv = c$ suppose that $a = 2$ and $b = 1$. Then to find by the alignment device (§ 220) the value of the sum $au + bv$ on the mid-axis, called the *axis of support*, it is necessary that equal distances above 0 on the outer axis A shall represent the same number, and similarly for the other outer axis B.

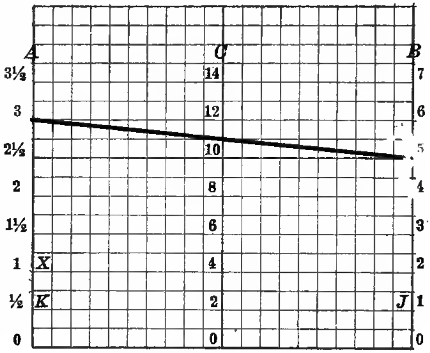


Chart of $2u + v = c$

Since $a = 2$ and $b = 1$, and since $2 \times \frac{1}{2} = 1$ and $1 \times 1 = 1$, u must be $\frac{1}{2}$ at the point K, and v must be 1 at the point J, and the point $2u$ must be on a level with v.

If we wish to plot different values of u and v so that we can find, for instance, $2 \times 3 + 1 \times 5$, by the alignment-chart method, we must make the scale of u on A twice as long as the scale of v on B, as indicated by the inner figures above. Thus, to find $2 \times 3 + 1 \times 5$, join $u = 3$ to $v = 5$ as shown on the figure. The line then passes through $c = 11$ on C.

If the equation is $3u + 2v = c$, since $3u = 1$ must always be on a level with $2v = 1$, it follows that the scale of u must be only $\frac{3}{2}$ of the scale of v. We may express this by saying that u scale : v scale $= 3 : 2$. If the equation is $5u + 7v = c$, the u scale is $\frac{5}{7}$ of the v scale.

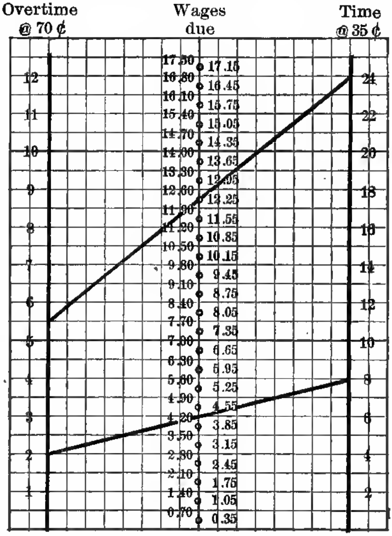
In general, in the equation $au + bv = c$ the product au must be equal to the product bv in order that equal distances from the zero points on A and B may represent equal amounts. This is necessary if we adhere to the simple addition principle explained above.

222. Wage Alignment Chart. Suppose that workmen are paid 35¢ an hour for regular time and 70¢ an hour for overtime work. The wage equation is then

$$70\,u + 35\,v = c,$$

where u represents various numbers of hours at the overtime rate, v the hours at the regular time rate, and c the total wages due the workman. By the principle of § 221 the scale of u must be twice as long as the scale of v.

The following is a portion of an alignment chart for wages at 35¢ per hour for regular time and 70¢ per hour for overtime:



The limits of space permit of showing only a small portion of an alignment chart, and this much reduced in size. It is, however, sufficient for our purposes. The wages due are shown on the central axis, or axis of support.

To find the wages due for 24 hr. time and $5\frac{1}{2}$ hr. overtime, connect these numbers as shown on the chart and read the value of c, which is $12.25.

Similarly, to find the wages for 8 hr. time and 2 hr. overtime, we have $4.20.

Any combination of time and overtime hours, the wages for which amount to $4.20, is represented on the outer axes. If the points representing these numbers are connected by a straight line as in the figure, this line will evidently pass through the same point on the axis of support as the line joining 2 and 8 in the figure.

The student should prove that a line drawn from 4, on the overtime axis, through $4.20 on the axis of wages due will meet the regular-time axis at 4.

Exercise 90. Construction and Use of Charts

1. Construct an alignment chart for adding any two numbers within the limits 250 to 266, inclusive.

The outer axes start at 250, not at 0. On these two axes use a scale of $\frac{1}{2}$ in. to a unit.

2. In Ex. 1 take the limits 300 and 315.

3. Construct an alignment chart for adding numbers ranging from 60 to 80 on one axis and from 50 to 70 on the other axis.

4. Construct a paymaster's alignment chart for workmen's wages, the scale being 50¢ an hour up to 15 hr. for regular time and 75¢ an hour up to 10 hr. for overtime.

5. Construct a chart similar to that in Ex. 4, and fill out the column of wages due in the following payroll:

WEEKLY PAYROLL OF THE ATLANTIC SUPPLY COMPANY

No.	Name	Hours @ 50¢	Hours @ 75¢	Wages	
1	Benson, A.	40	6		
2	Collins, John	39	2		
3	Crosby, H.	44			
4	Edelman, R.	42	3		
5	Epstein, M.	44	4		
6	Fish, J. R.	41	2		
7	Hobbs, Chas.	44			
8	King, M. N.	38			
9	Landsman, P.	40			
10	Smith, P. D.	44	4		
11	Turner, S. M.	44	8		
12	White, Jos.	44	3		

223. Logarithmic Chart. Any equation that can be reduced to the form

$$au + bv = c$$

can be made the subject of an alignment chart for the purposes of rapid calculation. For example, given the equation

$$xy = K,$$

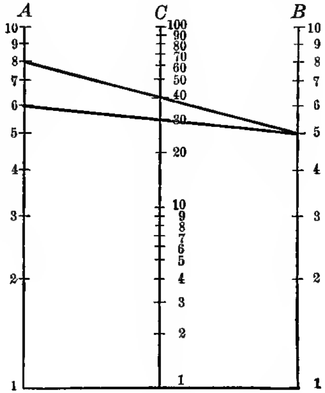
it follows that $\log x + \log y = \log K.$

This logarithmic equation is in the form

$$au + bv = c,$$

where $a = 1,\ b = 1,\ u = \log x,\ v = \log y,$ and $c = \log K.$

We may lay off lengths representing the mantissas of logarithms as we did in constructing slide rules, using the same scale on the axes A and B. The line joining corresponding points on A and B will strike points on C whose ordinates, or distances from the line 111, will represent the logarithm of the product. By connecting a few corresponding points we can easily find the scale to be used on C.



That is, a line from 2 to 2 must cross C at 4, and one from 3 to 3 must cross C at 9, and so on.

The small chart here shown contains only a few numbers, closely arranged, but a larger chart would give very satisfactory results.

If we wish the product of 5 and 8 we observe that the line crosses C at 40, and similarly we see that $5 \times 6 = 30$.

To divide 30 by 6 we draw a line from 6 on A to 30 on C and produce it to cut B, the point thus found being 5.

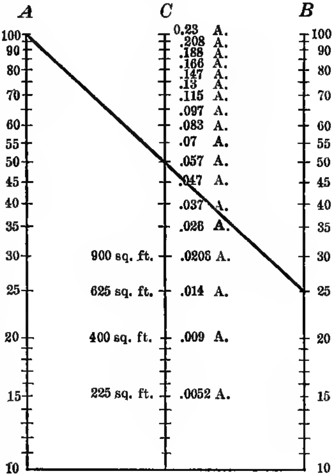
224. Chart for $z = xy/c$, where c is a Constant. Suppose that a real-estate dealer sells lots 100 ft. deep and of various frontages. Since there are 43,560 sq. ft. in an acre, the formula needed for finding the number of acres is $A = lw/43{,}560$, where A is the number of acres, l the length in feet, and w the width in feet.

Therefore $$\log A = \log l + \log w - \log 43{,}560.$$

Transposing, $\log l + \log w = \log A + \log 43{,}560.$

Since l and w are the variable numbers which when multiplied together must give the constant result A, we see that $\log A + \log 43{,}560$ is a constant. This reduces to the form of the preceding paragraph.

Using the same logarithmic scale for the A and B axes, we connect points on the same level, getting the number of square feet. The scale on C is half as long as that on A or B. The square feet can be expressed as acres by the above formula. If the principal points of division are indicated on C, the chart is more convenient for use than a table of values.



Exercise 91. Logarithmic Alignment Charts

1. Construct a logarithmic alignment chart for finding the number of square feet in a rectangle when the length and width are given in inches.

2. Construct a logarithmic alignment chart for use in the calculations of $pv = c$, the formula for Boyle's Law.

See the chart in § 223.

Exercise 92. Review of Chapter XII

1. Construct an alignment chart for adding any two numbers, of which one number falls between 590 and 620 and the other falls between 735 and 765.

2. Explain how to use the chart in Ex. 1 for subtraction, and state the limits within which it is available.

3. Explain by the aid of a chart the ratio of scale lengths on the axes of regular time and overtime necessary to construct a wage chart when the time rate is 70¢ per hour and the overtime rate is $1.05.

4. In the logarithmic alignment charts which we have constructed, the scale on the mid-axis was half as long as the scale on the side axes. Explain the manner in which an ordinary 10-inch slide rule may be used to mark the scale divisions in this way on the three axes.

5. If we wish to use an alignment chart of the type related to the equation $xy = K$, for the purpose of finding the squares of numbers, how do we proceed? Illustrate by a rough chart.

6. Insert a fourth axis D on a chart made like the one in § 223, and mark D so as to allow of finding the products of the numbers on B and C respectively. The scale length on D will be what part of the scale length on C? Why is it that a 10-inch and a 5-inch slide rule will enable you to mark all these scales?

7. A photographer announces that he will print and finish pictures from films or plates at a certain price per dozen when the negative has an area less than $\frac{1}{2}$ sq. ft., and at a price amounting to 25¢ more per dozen for each additional $\frac{1}{4}$ sq. ft. or a fraction thereof. Since the dimensions of the negatives are given in inches, he wishes a device that will easily give the area in square feet. Construct such a device.

Exercise 93. Review of Chapters I–XII

1. By the aid of logarithms calculate the price item for a bond table showing the price to be paid for a $1000 bond bearing 6%, having $3\frac{1}{2}$ yr. to run, to realize 5.45%.

2. By the aid of the Binomial Theorem find the compound amount of 1 for 6 yr. at 2%.

3. A man left $500,000 to a college on condition that the trustees should invest it until the accumulated amount was $600,000, at which time they were to use it for a designated purpose. If the fund accumulated at 4% compound interest, payable annually, how long before the fund was available?

4. A railway company wishes to float a bond issue bearing 5%, redeemable in 20 yr., interest payable semiannually, and to provide a sinking fund which it estimated can be accumulated at 4%, interest accumulated semiannually. If it can set aside $200,000 annually out of its net income to provide for interest and sinking fund, what is the total amount, to the next lower $1000, that it can borrow?

5. If the bond issue in Ex. 4 is to be in denominations of $1000, how many bonds would the company issue, and how much of the $200,000 per annum would remain unused?

6. A corporation wishes to borrow $300,000 on notes maturing in 1 yr., 2 yr., 3 yr., and 4 yr. respectively, the notes bearing 5% interest. If the corporation wishes to equalize the total annual burdens for interest and notes retired, find the face of each note.

7. Find the rate of yield on an investment in which a man pays $100 annually and receives $1400 at the end of 11 yr.

For the method of proceeding consult § 218.

8. What is the compound amount of $900 in $8\frac{1}{2}$ yr. at $4\frac{1}{4}$%, compounded annually?

TABLES

225. Explanation of the Tables. These seven tables are given in the following pages:

I. *Compound Amount on 1*, pages 210–213, expressing the value of s in the formula $s=(1+i)^n$ for the values of n from 1 to 100 and for the values $i=1\frac{1}{4}\%$, $1\frac{1}{2}\%$, 2%, $2\frac{1}{2}\%$, 3%, $3\frac{1}{2}\%$, 4%, $4\frac{1}{2}\%$, 5%, 6%.

II. *Present Value of 1*, pages 214–217, expressing the value of v^n in the formula $v^n=(1+i)^{-n}$ for the values of n and i as in Table I.

III. *The Amount of 1 per Annum*, pages 218–221, expressing the value of $s_{\overline{n}|}$ in the formula $s_{\overline{n}|}=[(1+i)^n-1]/i$ for the values of n and i as in Table I.

IV. *The Present Value of 1 per Annum*, pages 222–225, expressing the value of $a_{\overline{n}|}=(1-v^n)/i$ for the values of n and i as in Table I.

V. *The Annuity that 1 will Purchase*, pages 226–229, expressing the value of $1/a_{\overline{n}|}$ in the formula $1/a_{\overline{n}|}=i/(1-v^n)$ or $1/s_{\overline{n}|}=1/a_{\overline{n}|}-i$, for the values of n and i as in Table I.

The above tables are reproduced from the Mathematical Tables of Professor E. B. Skinner (Ginn and Company) by the kind permission of the author and the publishers.

VI. *Powers and Roots*, page 230, giving the squares, cubes, square roots, and cube roots of all integers from 1 to 100.

VII. *Logarithms*, pages 231–248, giving the logarithms to six places of all integral numbers from 100 to 9999.

I. COMPOUND AMOUNT ON 1; $s = (1 + i)^n$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	1.025 1563	1.030 2250	1.040 4000	1.050 6250	1.060 9000	2
3	1.037 9707	1.045 6784	1.061 2080	1.076 8906	1.092 7270	3
4	1.050 9453	1.061 3636	1.082 4322	1.103 8129	1.125 5088	4
5	1.064 0822	1.077 2840	1.104 0808	1.131 4082	1.159 2741	5
6	1.077 3832	1.093 4433	1.126 1624	1.159 6934	1.194 0523	6
7	1.090 8505	1.109 8449	1.148 6857	1.188 6858	1.229 8739	7
8	1.104 4861	1.126 4926	1.171 6594	1.218 4029	1.266 7701	8
9	1.118 2922	1.143 3900	1.195 0926	1.248 8630	1.304 7732	9
10	1.132 2708	1.160 5408	1.218 9944	1.280 0845	1.343 9164	10
11	1.146 4242	1.177 9489	1.243 3743	1.312 0867	1.384 2339	11
12	1.160 7545	1.195 6182	1.268 2418	1.344 8888	1.425 7609	12
13	1.175 2639	1.213 5524	1.293 6066	1.378 5110	1.468 5337	13
14	1.189 9547	1.231 7557	1.319 4788	1.412 9738	1.512 5897	14
16	1.204 8292	1.250 2321	1.345 8683	1.448 2982	1.557 9674	15
16	1.219 8895	1.268 9855	1.372 7857	1.484 5056	1.604 7064	16
17	1.235 1382	1.288 0203	1.400 2414	1.521 6183	1.652 8476	17
18	1.250 5774	1.307 3406	1.428 2463	1.559 6587	1.702 4331	18
19	1.266 2096	1.326 9507	1.456 8112	1.598 6502	1.753 5061	19
20	1.282 0372	1.346 8550	1.485 9474	1.638 6164	1.806 1112	20
21	1.298 0627	1.367 0578	1.515 6663	1.679 5819	1.860 2946	21
22	1.314 2885	1.387 5637	1.545 9797	1.721 5714	1.916 1034	22
23	1.330 7171	1.408 3771	1.576 8993	1.764 6107	1.973 5865	23
24	1.347 3511	1.429 5028	1.608 4373	1.808 7260	2.032 7941	24
25	1.364 1929	1.450 9454	1.640 6060	1.853 9441	2.093 7780	25
26	1.381 2454	1.472 7095	1.673 4181	1.900 2927	2.156 5913	26
27	1.398 5109	1.494 8002	1.706 8865	1.947 8000	2.221 2890	27
28	1.415 9923	1.517 2222	1.741 0242	1.996 4950	2.287 9277	28
29	1.433 6922	1.539 9805	1.775 8447	2.046 4074	2.356 5655	29
30	1.451 6134	1.563 0802	1.811 3616	2.097 5676	2.427 2625	30
31	1.469 7585	1.586 5264	1.847 5888	2.150 0068	2.500 0803	31
32	1.488 1305	1.610 3243	1.884 5406	2.203 7569	2.575 0828	32
33	1.506 7321	1.634 4792	1.922 2314	2.258 8509	2.652 3352	33
34	1.525 5663	1.658 9964	1.960 6760	2.315 3221	2.731 9053	34
35	1.544 6359	1.683 8813	1.999 8896	2.373 2052	2.813 8625	35
36	1.563 9438	1.709 1395	2.039 8873	2.432 5353	2.898 2783	36
37	1.583 4931	1.734 7766	2.080 6851	2.493 3487	2.985 2267	37
38	1.603 2868	1.760 7983	2.122 2988	2.555 6824	3.074 7835	38
39	1.623 3279	1.787 2102	2.164 7448	2.619 5745	3.167 0270	39
40	1.643 6195	1.814 0184	2.208 0397	2.685 0638	3.262 0378	40
41	1.664 1647	1.841 2287	2.252 2005	2.752 1904	3.359 8989	41
42	1.684 9668	1.868 8471	2.297 2445	2.820 9952	3.460 6959	42
43	1.706 0289	1.896 8798	2.343 1894	2.891 5201	3.564 5168	43
44	1.727 3542	1.925 3330	2.390 0531	2.963 8081	3.671 4523	44
45	1.748 9461	1.954 2130	2.437 8542	3.037 9033	3.781 5958	45
46	1.770 8080	1.983 5262	2.486 6113	3.113 8509	3.895 0437	46
47	1.792 9431	2.013 2791	2.536 3435	3.191 6971	4.011 8950	47
48	1.815 3549	2.043 4783	2.587 0704	3.271 4896	4.132 2519	48
49	1.838 0468	2.074 1305	2.638 8118	3.353 2768	4.256 2194	49
50	1.861 0224	2.105 2424	2.691 5880	3.437 1087	4.383 9060	50

I. COMPOUND AMOUNT ON 1; $s = (1 + i)^n$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	1.884 2851	2.136 8211	2.745 4198	3.523 0364	4.515 4232	**51**
52	1.907 8387	2.168 8734	2.800 3282	3.611 1123	4.650 8859	**52**
53	1.931 6867	2.201 4065	2.856 3347	3.701 3902	4.790 4125	**53**
54	1.955 8328	2.234 4276	2.913 4614	3.793 9249	4.934 1248	**54**
55	1.980 2807	2.267 9440	2.971 7307	3.888 7730	5.082 1486	**55**
56	2.005 0342	2.301 9631	3.031 1653	3.985 9924	5.234 6130	**56**
57	2.030 0971	2.336 4926	3.091 7886	4.085 6422	5.391 6514	**57**
58	2.055 4733	2.371 5400	3.156 6244	4.187 7832	5.553 4010	**58**
59	2.081 1668	2.407 1131	3.216 6969	4.292 4778	5.720 0030	**59**
60	2.107 1884	2.443 2198	3.281 0308	4.399 7897	5.891 6031	**60**
61	2.133 5211	2.479 8681	3.346 6514	4.509 7845	6.068 3512	**61**
62	2.160 1901	2.517 0661	3.413 5844	4.622 5291	6.250 4017	**62**
63	2.187 1925	2.554 8221	3.481 8561	4.738 0923	6.437 9138	**63**
64	2.214 5324	2.593 1444	3.551 4932	4.856 5446	6.631 0512	**64**
65	2.242 2141	2.632 0416	3.622 5231	4.977 9583	6.829 9827	**65**
66	2.270 2417	2.671 5222	3.694 9736	5.102 4072	7.034 8822	**66**
67	2.298 6198	2.711 5950	3.768 8730	5.229 9674	7.245 9287	**67**
68	2.327 3525	2.752 2690	3.844 2505	5.360 7166	7.463 3065	**68**
69	2.356 4444	2.793 5530	3.921 1355	5.494 7345	7.687 2057	**69**
70	2.385 9000	2.835 4563	3.999 5582	5.632 1029	7.917 8219	**70**
71	2.415 7237	2.877 9881	4.079 5494	5.772 9054	8.155 3566	**71**
72	2.445 9203	2.921 1580	4.161 1404	5.917 2281	8.400 0173	**72**
73	2.476 4943	2.964 9753	4.244 3632	6.065 1588	8.652 0178	**73**
74	2.507 4505	3.009 4500	4.329 2504	6.216 7877	8.911 5783	**74**
75	2.538 7936	3.054 5917	4.415 8355	6.372 2074	9.178 9257	**75**
76	2.570 5285	3.100 4106	4.504 1522	6.531 5126	9.454 2934	**76**
77	2.602 6601	3.146 9167	4.594 2352	6.694 8004	9.737 9222	**77**
78	2.635 1934	3.194 1205	4.686 1199	6.862 1704	10.030 0599	**78**
79	2.668 1333	3.242 0323	4.779 8423	7.033 7247	10.330 9617	**79**
80	2.701 4849	3.290 6628	4.875 4392	7.209 5678	10.640 8906	**80**
81	2.735 2535	3.340 0227	4.972 9479	7.389 8070	10.960 1173	**81**
82	2.769 4442	3.390 1231	5.072 4069	7.574 5522	11.288 9208	**82**
83	2.804 0622	3.440 9749	5.173 8550	7.763 9160	11.627 5884	**83**
84	2.839 1130	3.492 5895	5.277 3321	7.958 0139	11.976 4161	**84**
85	2.874 6019	3.544 9784	5.382 8788	8.156 9642	12.335 7085	**85**
86	2.910 5344	3.598 1531	5.490 5364	8.360 8883	12.705 7798	**86**
87	2.946 9161	3.652 1253	5.600 3471	8.569 9106	13.086 9532	**87**
88	2.983 7526	3.706 9072	5.712 3540	8.784 1583	13.479 5618	**88**
89	3.021 0495	3.762 5108	5.826 6011	9.003 7623	13.883 9487	**89**
90	3.058 8126	3.818 9485	5.943 1331	9.228 8563	14.300 4671	**90**
91	3.097 0477	3.876 2327	6.061 9958	9.459 5777	14.729 4811	**91**
92	3.135 7609	3.934 3762	6.183 2357	9.696 0672	15.171 3656	**92**
93	3.174 9579	3.993 3919	6.306 9004	9.938 4689	15.626 5065	**93**
94	3.214 6448	4.053 2927	6.433 0384	10.186 9306	16.095 3017	**94**
95	3.254 8279	4.114 0921	6.561 6992	10.441 6038	16.578 1608	**95**
96	3.295 5132	4.175 8035	6.692 9332	10.702 6439	17.075 5056	**96**
97	3.336 7072	4.238 4406	6.826 7918	10.970 2100	17.587 7708	**97**
98	3.378 4160	4.302 0172	6.963 3277	11.244 4653	18.115 4039	**98**
99	3.420 6462	4.366 5474	7.102 5942	11.525 5769	18.658 8660	**99**
100	3.463 4043	4.432 0457	7.244 6461	11.813 7164	19.218 6320	**100**

I. COMPOUND AMOUNT ON 1; $s = (1 + i)^n$

n	$3\frac{1}{2}$%	4%	$4\frac{1}{2}$%	5%	6%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	**1**
2	1.071 2250	1.081 6000	1.092 0250	1.102 5000	1.123 6000	**2**
3	1.108 7179	1.124 8640	1.141 1661	1.157 6250	1.191 0160	**3**
4	1.147 5230	1.169 8586	1.192 5186	1.215 5063	1.262 4770	**4**
5	1.187 6863	1.216 6529	1.246 1819	1.276 2816	1.338 2256	**5**
6	1.229 2553	1.265 3190	1.302 2601	1.340 0956	1.418 5191	**6**
7	1.272 2793	1.315 9318	1.360 8618	1.407 1004	1.503 6303	**7**
8	1.316 8090	1.368 5691	1.422 1006	1.477 4554	1.593 8481	**8**
9	1.362 8974	1.423 3118	1.486 0951	1.551 3282	1.689 4790	**9**
10	1.410 5988	1.480 2443	1.552 9694	1.628 8946	1.790 8477	**10**
11	1.459 9697	1.539 4541	1.622 8530	1.710 3394	1.898 2986	**11**
12	1.511 0687	1.601 0322	1.695 8814	1.795 8563	2.012 1965	**12**
13	1.563 9561	1.665 0735	1.772 1961	1.885 6491	2.132 9283	**13**
14	1.618 6945	1.731 6764	1.851 9449	1.979 9316	2.260 9040	**14**
15	1.675 3488	1.800 9435	1.935 2824	2.078 9282	2.396 5582	**15**
16	1.733 9860	1.872 9812	2.022 3702	2.182 8746	2.540 3517	**16**
17	1.794 6756	1.947 9005	2.113 3768	2.292 0183	2.692 7728	**17**
18	1.857 4892	2.025 8165	2.208 4788	2.406 6192	2.854 3392	**18**
19	1.922 5013	2.106 8492	2.307 8603	2.526 9502	3.025 5995	**19**
20	1.989 7889	2.191 1231	2.411 7140	2.653 2977	3.207 1355	**20**
21	2.059 4315	2.278 7681	2.520 2412	2.785 9626	3.399 5636	**21**
22	2.131 5116	2.369 9188	2.633 6520	2.925 2607	3.603 5374	**22**
23	2.206 1145	2.464 7155	2.752 1663	3.071 5238	3.819 7497	**23**
24	2.283 3285	2.563 3042	2.876 0138	3.225 0999	4.048 9346	**24**
25	2.363 2450	2.665 8363	3.005 4345	3.386 3549	4.291 8707	**25**
26	2.445 9586	2.772 4698	3.140 6790	3.555 6727	4.549 3830	**26**
27	2.531 5671	2.883 3686	3.282 0096	3.733 4563	4.822 3459	**27**
28	2.620 1720	2.998 7033	3.429 7000	3.920 1291	5.111 6867	**28**
29	2.711 8780	3.118 6515	3.584 0365	4.116 1356	5.418 3879	**29**
30	2.806 7937	3.243 3975	3.745 3181	4.321 9424	5.743 4912	**30**
31	2.905 0315	3.373 1334	3.913 8575	4.538 0395	6.088 1006	**31**
32	3.006 7076	3.508 0587	4.089 9810	4.764 9415	6.453 3867	**32**
33	3.111 9424	3.648 3811	4.274 0302	5.003 1885	6.840 5899	**33**
34	3.220 8603	3.794 3163	4.466 3615	5.253 3480	7.251 0253	**34**
35	3.333 5904	3.946 0890	4.667 3478	5.516 0154	7.686 0868	**35**
36	3.450 2661	4.103 9326	4.877 3785	5.791 8161	8.147 2520	**36**
37	3.571 0254	4.268 0899	5.096 8605	6.081 4069	8.636 0871	**37**
38	3.696 0113	4.438 8135	5.326 2192	6.385 4773	9.154 2523	**38**
39	3.825 3717	4.616 3660	5.565 8991	6.704 7512	9.703 5075	**39**
40	3.959 2597	4.801 0206	5.816 3645	7.039 9887	10.285 7179	**40**
41	4.097 8338	4.993 0615	6.078 1009	7.391 9881	10.902 8610	**41**
42	4.241 2580	5.192 7839	6.351 6155	7.761 5876	11.557 0327	**42**
43	4.389 7020	5.400 4953	6.637 4382	8.149 6669	12.250 4546	**43**
44	4.543 3416	5.616 5151	6.936 1229	8.557 1503	12.985 4819	**44**
45	4.702 3586	5.841 1757	7.248 2484	8.985 0078	13.764 6108	**45**
46	4.866 9411	6.074 8227	7.574 4196	9.434 2582	14.590 4875	**46**
47	5.037 2840	6.317 8156	7.915 2685	9.905 9711	15.465 9167	**47**
48	5.213 5890	6.570 5282	8.271 4556	10.401 2696	16.393 8717	**48**
49	5.396 0646	6.833 3494	8.643 6711	10.921 3331	17.377 5040	**49**
50	5.584 9269	7.106 6833	9.032 6363	11.467 3998	18.420 1543	**50**

I. COMPOUND AMOUNT ON 1; $s = (1 + i)^n$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	5.780 3993	7.390 9507	9.439 1049	12.040 7698	19.525 3635	**51**
52	5.982 7133	7.686 5887	9.863 8646	12.642 8083	20.696 8853	**52**
53	6.192 1082	7.994 0523	10.307 7385	13.274 9487	21.938 6985	**53**
54	6.408 8320	8.313 8143	10.771 5868	13.938 6961	23.255 0204	**54**
55	6.633 1411	8.646 3669	11.256 3082	14.635 6309	24.650 3216	**55**
56	6.865 3011	8.992 2216	11.762 8420	15.367 4125	26.129 3409	**56**
57	7.105 5866	9.351 9105	12.292 1699	16.135 7831	27.697 1013	**57**
58	7.354 2822	9.725 9869	12.845 3176	16.942 5722	29.358 9274	**58**
59	7.611 6820	10.115 0264	13.423 3569	17.789 7009	31.120 4631	**59**
60	7.878 0909	10.519 6274	14.027 4079	18.679 1859	32.987 6909	**60**
61	8.153 8241	10.940 4125	14.658 6413	19.613 1452	34.966 9523	**61**
62	8.439 2079	11.378 0290	15.318 2801	20.593 8025	37.064 9694	**62**
63	8.734 5802	11.833 1502	16.007 6028	21.623 4926	39.288 8676	**63**
64	9.040 2905	12.306 4762	16.727 9449	22.704 6672	41.646 1997	**64**
65	9.356 7007	12.798 7352	17.480 7024	23.839 9006	44.144 9717	**65**
66	9.684 1852	13.310 6846	18.267 3340	25.031 8956	46.793 6699	**66**
67	10.023 1317	13.843 1120	19.089 3640	26.283 4904	49.601 2901	**67**
68	10.373 9413	14.396 8365	19.948 3854	27.597 6649	52.577 3676	**68**
69	10.737 0292	14.972 7100	20.846 0628	28.977 5481	55.732 0096	**69**
70	11.112 8253	15.571 6184	21.784 1356	30.426 4255	59.075 9302	**70**
71	11.501 7741	16.194 4831	22.764 4217	31.947 7468	62.620 4860	**71**
72	11.904 3362	16.842 2624	23.788 8207	33.545 1342	66.377 7152	**72**
73	12.320 9880	17.515 9529	24.859 3176	35.222 3909	70.360 3781	**73**
74	12.752 2226	18.216 5910	25.977 9869	36.983 5104	74.582 0007	**74**
75	13.198 5504	18.945 2547	27.146 9963	38.832 6859	79.056 9208	**75**
76	13.660 4996	19.703 0648	28.368 6111	40.774 3202	83.800 3360	**76**
77	14.138 6171	20.491 1874	29.645 1986	42.813 0362	88.828 3562	**77**
78	14.633 4687	21.310 8349	30.979 2326	44.953 6880	94.158 0576	**78**
79	15.145 6401	22.163 2683	32.373 2980	47.201 3724	99.807 5410	**79**
80	15.675 7375	23.049 7991	33.830 0964	49.561 4411	105.795 9935	**80**
81	16.224 3884	23.971 7910	35.352 4508	52.039 5131	112.143 7531	**81**
82	16.792 2419	24.930 6627	36.943 3111	54.641 4888	118.872 3782	**82**
83	17.379 9704	25.927 8892	38.605 7601	57.373 5632	126.004 7210	**83**
84	17.988 2694	26.965 0047	40.343 0193	60.242 2414	133.565 0042	**84**
85	18.617 8588	28.043 6049	42.158 4551	63.254 3534	141.578 9045	**85**
86	19.269 4839	29.165 3491	44.055 5856	66.417 0711	150.073 6388	**86**
87	19.943 9158	30.331 9631	46.038 0870	69.737 9247	159.078 0571	**87**
88	20.641 9529	31.545 2416	48.109 8009	73.224 8209	168.622 7405	**88**
89	21.364 4212	32.807 0513	50.274 7419	76.886 0620	178.740 1049	**89**
90	22.112 1759	34.119 3333	52.537 1053	80.730 3651	189.464 5112	**90**
91	22.886 1021	35.484 1067	54.901 2750	84.766 8833	200.832 3819	**91**
92	23.687 1157	36.903 4709	57.371 8324	89.005 2275	212.882 3248	**92**
93	24.516 1647	38.379 6098	59.953 5649	93.455 4888	225.655 2643	**93**
94	25.374 2305	39.914 7942	62.651 4753	98.128 2633	239.194 5801	**94**
95	26.262 3286	41.511 3859	65.470 7917	103.034 6765	253.546 2550	**95**
96	27.181 5101	43.171 8414	68.416 9773	108.186 4103	268.759 0303	**96**
97	28.132 8629	44.898 7150	71.495 7413	113.595 7308	284.884 5721	**97**
98	29.117 5131	46.694 6636	74.713 0496	119.275 5173	301.977 6464	**98**
99	30.136 6261	48.562 4502	78.075 1369	125.239 2932	320.096 3052	**99**
100	31.191 4080	50.504 9482	81.588 5180	131.501 2579	339.302 0835	**100**

II. PRESENT VALUE OF 1; $v^n = (1 + i)^{-n}$

n	$1\frac{1}{4}$ %	$1\frac{1}{2}$ %	2 %	$2\frac{1}{2}$ %	3 %	**n**
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	**1**
2	0.975 4611	0.970 6617	0.961 1688	0.951 8144	0.942 5959	**2**
3	0.963 4183	0.956 3170	0.942 3223	0.928 5994	0.915 1417	**3**
4	0.951 5243	0.942 1842	0.923 8454	0.905 9506	0.888 4870	**4**
5	0.939 7771	0.928 2603	0.905 7308	0.883 8543	0.862 6088	**5**
6	0.928 1749	0.914 5422	0.887 9714	0.862 2969	0.837 4843	**6**
7	0.916 7159	0.901 0268	0.870 5602	0.841 2652	0.813 0915	**7**
8	0.905 3985	0.887 7111	0.853 4904	0.820 7466	0.789 4092	**8**
9	0.894 2207	0.874 5922	0.836 7553	0.800 7284	0.766 4167	**9**
10	0.883 1809	0.861 6672	0.820 3483	0.781 1984	0.744 0939	**10**
11	0.872 2775	0.848 9332	0.804 2630	0.762 1448	0.722 4213	**11**
12	0.861 5086	0.836 3874	0.788 4932	0.743 5559	0.701 3799	**12**
13	0.850 8727	0.824 0270	0.773 0325	0.725 4204	0.680 9513	**13**
14	0.840 3681	0.811 8493	0.757 8750	0.707 7272	0.661 1178	**14**
15	0.829 9932	0.799 8515	0.743 0147	0.690 4656	0.641 8619	**15**
16	0.819 7463	0.788 0310	0.728 4458	0.673 6249	0.623 1669	**16**
17	0.809 6260	0.776 3853	0.714 1626	0.657 1951	0.605 0164	**17**
18	0.799 6306	0.764 9116	0.700 1594	0.641 1659	0.587 3946	**18**
19	0.789 7587	0.756 6075	0.686 4308	0.625 5277	0.570 2860	**19**
20	0.780 0085	0.742 4704	0.672 9713	0.610 2709	0.553 6758	**20**
21	0.770 3788	0.731 4980	0.659 7758	0.595 3863	0.537 5493	**21**
22	0.760 8680	0.720 6876	0.646 8390	0.580 8647	0.521 8925	**22**
23	0.751 4745	0.710 0371	0.634 1559	0.566 6972	0.506 6917	**23**
24	0.742 1971	0.699 5439	0.621 7215	0.552 8754	0.491 9337	**24**
25	0.733 0341	0.689 2058	0.609 5309	0.539 3906	0.477 6056	**25**
26	0.723 9843	0.679 0205	0.597 5793	0.526 2347	0.463 6947	**26**
27	0.715 0463	0.668 9857	0.585 8620	0.513 3997	0.450 1891	**27**
28	0.706 2185	0.659 0993	0.574 3746	0.500 8778	0.437 0768	**28**
29	0.697 4998	0.649 3589	0.563 1123	0.488 6613	0.424 3464	**29**
30	0.688 8887	0.639 7624	0.552 0709	0.476 7427	0.411 9868	**30**
31	0.680 3839	0.630 3078	0.541 2460	0.465 1148	0.399 9871	**31**
32	0.671 9841	0.620 9929	0.530 6333	0.453 7705	0.388 3370	**32**
33	0.663 6880	0.611 8157	0.520 2287	0.442 7030	0.377 0262	**33**
34	0.655 4943	0.602 7741	0.510 0282	0.431 9053	0.366 0449	**34**
35	0.647 4018	0.593 8661	0.500 0276	0.421 3711	0.355 3834	**35**
36	0.639 4092	0.585 0897	0.490 2232	0.411 0937	0.345 0324	**36**
37	0.631 5152	0.576 4431	0.480 6109	0.401 0670	0.334 9829	**37**
38	0.623 7187	0.567 9242	0.471 1872	0.391 2849	0.325 2262	**38**
39	0.616 0185	0.559 5313	0.461 9482	0.381 7414	0.315 7535	**39**
40	0.608 4133	0.551 2623	0.452 8904	0.372 4306	0.306 5568	**40**
41	0.600 9021	0.543 1156	0.444 0102	0.363 3469	0.297 6280	**41**
42	0.593 4835	0.535 0893	0.435 3041	0.354 4848	0.288 9592	**42**
43	0.586 1566	0.527 1815	0.426 7688	0.345 8389	0.280 5429	**43**
44	0.578 9201	0.519 3907	0.418 4007	0.337 4038	0.272 3718	**44**
45	0.571 7729	0.511 7149	0.410 1968	0.329 1744	0.264 4386	**45**
46	0.564 7140	0.504 1527	0.402 1537	0.321 1458	0.256 7365	**46**
47	0.557 7422	0.496 7021	0.394 2684	0.313 3129	0.249 2588	**47**
48	0.550 8565	0.489 3617	0.386 5376	0.305 6712	0.241 9988	**48**
49	0.544 0558	0.482 1298	0.378 9584	0.298 2158	0.234 9503	**49**
50	0.537 3391	0.475 0047	0.371 5279	0.290 9422	0.228 0171	**50**

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	0.530 7052	0.467 9849	0.364 2430	0.283 8461	0.221 4632	**51**
52	0.524 1533	0.461 0689	0.357 1010	0.276 9230	0.215 0128	**52**
53	0.517 6823	0.454 2550	0.350 0990	0.270 1688	0.208 7503	**53**
54	0.511 2912	0.447 5419	0.343 2343	0.263 5793	0.202 6702	**54**
55	0.504 9789	0.440 9280	0.336 5042	0.257 1505	0.196 7672	**55**
56	0.498 7446	0.434 4118	0.329 9061	0.250 8785	0.191 0361	**56**
57	0.492 5873	0.427 9919	0.323 4374	0.244 7596	0.185 4719	**57**
58	0.486 5059	0.421 6669	0.317 0955	0.238 7898	0.180 0698	**58**
59	0.480 4997	0.415 4354	0.310 8779	0.232 9657	0.174 8251	**59**
60	0.474 5676	0.409 2960	0.304 7823	0.227 2836	0.169 7331	**60**
61	0.468 7087	0.403 2473	0.298 8061	0.221 7401	0.164 7894	**61**
62	0.462 9222	0.397 2879	0.292 9472	0.216 3318	0.159 9897	**62**
63	0.457 2071	0.391 4167	0.287 2031	0.211 0554	0.155 3298	**63**
64	0.451 5626	0.385 6322	0.281 5717	0.205 9077	0.150 8057	**64**
65	0.445 9878	0.379 9332	0.276 0507	0.200 8856	0.146 4133	**65**
66	0.440 4817	0.374 3184	0.270 6379	0.195 9859	0.142 1488	**66**
67	0.435 0437	0.368 7866	0.265 3313	0.191 2058	0.138 0085	**67**
68	0.429 6728	0.363 3366	0.260 1287	0.186 5422	0.133 9889	**68**
69	0.424 3682	0.357 9671	0.255 0282	0.181 9924	0.130 0863	**69**
70	0.419 1291	0.352 6769	0.250 0276	0.177 5536	0.126 2974	**70**
71	0.413 9546	0.347 4650	0.245 1251	0.173 2230	0.122 6188	**71**
72	0.408 8441	0.342 3300	0.240 3187	0.168 9980	0.119 0474	**72**
73	0.403 7966	0.337 2709	0.235 6066	0.164 8761	0.115 5800	**73**
74	0.398 8115	0.332 2866	0.230 9869	0.160 8548	0.112 2136	**74**
75	0.393 8879	0.327 3760	0.226 4577	0.156 9315	0.108 9452	**75**
76	0.389 0251	0.322 5379	0.222 0174	0.153 1039	0.105 7721	**76**
77	0.384 2223	0.317 7714	0.217 6641	0.149 3697	0.102 6913	**77**
78	0.379 4788	0.313 0752	0.213 3962	0.145 7265	0.099 7003	**78**
79	0.374 7939	0.308 4485	0.209 2119	0.142 1722	0.096 7964	**79**
80	0.370 1668	0.303 8901	0.205 1097	0.138 7046	0.093 9771	**80**
81	0.365 5968	0.299 3992	0.201 0880	0.135 3215	0.091 2399	**81**
82	0.361 0833	0.294 9745	0.197 1451	0.132 0210	0.088 5824	**82**
83	0.356 6255	0.290 6153	0.193 2795	0.128 8010	0.086 0024	**83**
84	0.352 2227	0.286 3205	0.189 4897	0.125 6595	0.083 4974	**84**
85	0.347 8743	0.282 0892	0.185 7742	0.122 5946	0.081 0655	**85**
86	0.343 5795	0.277 9204	0.182 1316	0.119 6045	0.078 7043	**86**
87	0.339 3378	0.273 8132	0.178 5604	0.116 6873	0.076 4120	**87**
88	0.335 1484	0.269 7667	0.175 0592	0.113 8413	0.074 1864	**88**
89	0.331 0108	0.265 7800	0.171 6266	0.111 0647	0.072 0256	**89**
90	0.326 9242	0.261 8522	0.168 2614	0.108 3558	0.069 9278	**90**
91	0.322 8881	0.257 9824	0.164 9622	0.105 7130	0.067 8911	**91**
92	0.318 9019	0.254 1699	0.161 7276	0.103 1346	0.065 9136	**92**
93	0.314 9648	0.250 4137	0.158 5565	0.100 6191	0.063 9938	**93**
94	0.311 0764	0.246 7130	0.155 4475	0.098 1650	0.062 1299	**94**
95	0.307 2359	0.243 0670	0.152 3995	0.095 7707	0.060 3203	**95**
96	0.303 4429	0.239 4749	0.149 4113	0.093 4349	0.058 5634	**96**
97	0.299 6967	0.235 9358	0.146 4817	0.091 1560	0.056 8577	**97**
98	0.295 9967	0.232 4491	0.143 6095	0.088 9326	0.055 2016	**98**
99	0.292 3424	0.229 0139	0.140 7936	0.086 7635	0.053 5938	**99**
100	0.288 7333	0.225 6294	0.138 0330	0.084 6474	0.052 0328	**100**

II. PRESENT VALUE OF 1; $v^n = (1 + i)^{-n}$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	**1**
2	0.933 5107	0.924 5562	0.915 7300	0.907 0295	0.889 9964	**2**
3	0.901 9427	0.888 9964	0.876 2966	0.863 8376	0.839 6193	**3**
4	0.871 4422	0.854 8042	0.838 5613	0.822 7025	0.792 0937	**4**
5	0.841 9732	0.821 9271	0.802 4510	0.783 5262	0.747 2582	**5**
6	0.813 5006	0.790 3145	0.767 8957	0.746 2154	0.704 9605	**6**
7	0.785 9910	0.759 9178	0.734 8285	0.710 6813	0.665 0571	**7**
8	0.759 4116	0.730 6902	0.703 1851	0.676 8394	0.627 4124	**8**
9	0.733 7310	0.702 5867	0.672 9044	0.644 6089	0.591 8985	**9**
10	0.708 9188	0.675 5642	0.643 9277	0.613 9133	0.558 3948	**10**
11	0.684 9457	0.649 5809	0.616 1987	0.584 6793	0.526 7875	**11**
12	0.661 7833	0.624 5970	0.589 6639	0.556 8374	0.496 9694	**12**
13	0.639 4042	0.600 5741	0.564 2716	0.530 3214	0.468 8390	**13**
14	0.617 7818	0.577 4751	0.539 9729	0.505 0680	0.442 3010	**14**
15	0.596 8906	0.555 2645	0.516 7204	0.481 0171	0.417 2651	**15**
16	0.576 7059	0.533 9082	0.494 4693	0.458 1115	0.393 6463	**16**
17	0.557 2038	0.513 3732	0.473 1764	0.436 2967	0.371 3644	**17**
18	0.538 3611	0.493 6281	0.452 8004	0.415 5207	0.350 3438	**18**
19	0.520 1557	0.474 6424	0.433 3018	0.395 7340	0.330 5130	**19**
20	0.502 5659	0.456 3869	0.414 6429	0.376 8895	0.311 8047	**20**
21	0.485 5709	0.438 8336	0.396 7874	0.358 9424	0.294 1554	**21**
22	0.469 1506	0.421 9554	0.379 7009	0.341 8499	0.277 5051	**22**
23	0.453 2856	0.405 7263	0.363 3501	0.325 5713	0.261 7973	**23**
24	0.437 9571	0.390 1215	0.347 7035	0.310 0679	0.246 9785	**24**
25	0.423 1470	0.375 1168	0.332 7306	0.295 3028	0.232 9986	**25**
26	0.408 8377	0.360 6892	0.318 4025	0.281 2407	0.219 8100	**26**
27	0.395 0122	0.346 8166	0.304 6914	0.267 8483	0.207 3679	**27**
28	0.381 6543	0.333 4775	0.291 5707	0.255 0936	0.195 6301	**28**
29	0.368 7482	0.320 6514	0.279 0150	0.242 9463	0.184 5567	**29**
30	0.356 2784	0.308 3187	0.267 0000	0.231 3774	0.174 1101	**30**
31	0.344 2303	0.296 4603	0.255 5024	0.220 3595	0.164 2548	**31**
32	0.332 5897	0.285 0579	0.244 4999	0.209 8662	0.154 9574	**32**
33	0.321 3427	0.274 0942	0.233 9712	0.199 8725	0.146 1862	**33**
34	0.310 4761	0.263 5521	0.223 8959	0.190 3548	0.137 9115	**34**
35	0.299 9769	0.253 4155	0.214 2544	0.181 2903	0.130 1052	**35**
36	0.289 8327	0.243 6687	0.205 0282	0.172 6574	0.122 7408	**36**
37	0.280 0316	0.234 2968	0.196 1992	0.164 4356	0.115 7932	**37**
38	0.270 5619	0.225 2854	0.187 7504	0.156 6054	0.109 2389	**38**
39	0.261 4125	0.216 6206	0.179 6655	0.149 1480	0.103 0555	**39**
40	0.252 5725	0.208 2890	0.171 9287	0.142 0457	0.097 2222	**40**
41	0.244 0314	0.200 2779	0.164 5251	0.135 2816	0.091 7190	**41**
42	0.235 7791	0.192 5749	0.157 4403	0.128 8396	0.086 5274	**42**
43	0.227 8059	0.185 1682	0.150 6605	0.122 7044	0.081 6296	**43**
44	0.220 1023	0.178 0463	0.144 1728	0.116 8613	0.077 0091	**44**
45	0.212 6592	0.171 1984	0.137 9644	0.111 2965	0.072 6501	**45**
46	0.205 4679	0.164 6139	0.132 0233	0.105 9967	0.068 5378	**46**
47	0.198 5197	0.158 2826	0.126 3381	0.100 9492	0.064 6583	**47**
48	0.191 8065	0.152 1948	0.120 8977	0.096 1421	0.060 9984	**48**
49	0.185 3202	0.146 3411	0.115 6916	0.091 5639	0.057 5457	**49**
50	0.179 0534	0.140 7126	0.110 7096	0.087 2037	0.054 2884	**50**

II. PRESENT VALUE OF 1; $v^n = (1 + i)^{-n}$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	0.172 9984	0.135 3006	0.105 9422	0.083 0512	0.051 2154	**51**
52	0.167 1482	0.130 0967	0.101 3801	0.079 0964	0.048 3164	**52**
53	0.161 4959	0.125 0930	0.097 0145	0.075 3299	0.045 5816	**53**
54	0.156 0347	0.120 2817	0.092 8368	0.071 7427	0.043 0015	**54**
55	0.150 7581	0.115 6555	0.088 8391	0.068 3264	0.040 5674	**55**
56	0.145 6600	0.111 2072	0.085 0135	0.065 0728	0.038 2712	**56**
57	0.140 7343	0.106 9300	0.081 3526	0.061 9741	0.036 1049	**57**
58	0.135 9752	0.102 8173	0.077 8494	0.059 0229	0.034 0612	**58**
59	0.131 3770	0.098 8628	0.074 4970	0.056 2123	0.032 1332	**59**
60	0.126 9343	0.095 0604	0.071 2890	0.053 5355	0.030 3143	**60**
61	0.122 6418	0.091 4042	0.068 2191	0.050 9862	0.028 5984	**61**
62	0.118 4945	0.087 8887	0.065 2815	0.048 5583	0.026 9797	**62**
63	0.114 4875	0.084 5084	0.062 4703	0.046 2460	0.025 4525	**63**
64	0.110 6159	0.081 2580	0.059 7802	0.044 0438	0.024 0118	**64**
65	0.106 8753	0.078 1327	0.057 2059	0.041 9465	0.022 6526	**65**
66	0.103 2611	0.075 1276	0.054 7425	0.039 9490	0.021 3704	**66**
67	0.099 7692	0.072 2381	0.052 3852	0.038 0467	0.020 1608	**67**
68	0.096 3954	0.069 4597	0.050 1294	0.036 2349	0.019 0196	**68**
69	0.093 1356	0.066 7882	0.047 9707	0.034 5095	0.017 9430	**69**
70	0.089 9861	0.064 2194	0.045 9050	0.032 8662	0.016 9274	**70**
71	0.086 9431	0.061 7494	0.043 9282	0.031 3011	0.015 9692	**71**
72	0.084 0030	0.059 3744	0.042 0366	0.029 8106	0.015 0653	**72**
73	0.081 1623	0.057 0908	0.040 2264	0.028 3910	0.014 2125	**73**
74	0.078 4177	0.054 8950	0.038 4941	0.027 0391	0.013 4081	**74**
75	0.075 7659	0.052 7837	0.036 8365	0.025 7515	0.012 6491	**75**
76	0.073 2038	0.050 7535	0.035 2502	0.024 5252	0.011 9331	**76**
77	0.070 7283	0.048 8015	0.033 7323	0.023 3574	0.011 2577	**77**
78	0.068 3365	0.046 9245	0.032 2797	0.022 2451	0.010 6204	**78**
79	0.066 0256	0.045 1197	0.030 8897	0.021 1858	0.010 0193	**79**
80	0.063 7929	0.043 3843	0.029 5595	0.020 1770	0.009 4522	**80**
81	0.061 6356	0.041 7157	0.028 2866	0.019 2162	0.008 9171	**81**
82	0.059 5513	0.040 1112	0.027 0685	0.018 3011	0.008 4124	**82**
83	0.057 5375	0.038 5685	0.025 9029	0.017 4296	0.007 9362	**83**
84	0.055 5918	0.037 0851	0.024 7874	0.016 5996	0.007 4870	**84**
85	0.053 7119	0.035 6588	0.023 7200	0.015 8092	0.007 0632	**85**
86	0.051 8955	0.034 2873	0.022 6986	0.015 0564	0.006 6634	**86**
87	0.050 1406	0.032 9685	0.021 7211	0.014 3394	0.006 2862	**87**
88	0.048 4450	0.031 7005	0.020 7858	0.013 6566	0.005 9304	**88**
89	0.046 8068	0.030 4813	0.019 8907	0.013 0063	0.005 5947	**89**
90	0.045 2240	0.029 3089	0.019 0342	0.012 3869	0.005 2780	**90**
91	0.043 6946	0.028 1816	0.018 2145	0.011 7971	0.004 9793	**91**
92	0.042 2170	0.027 0977	0.017 4302	0.011 2353	0.004 6974	**92**
93	0.040 7894	0.026 0555	0.016 6796	0.010 7003	0.004 4315	**93**
94	0.039 4101	0.025 0534	0.015 9613	0.010 1907	0.004 1807	**94**
95	0.038 0774	0.024 0898	0.015 2740	0.009 7055	0.003 9441	**95**
96	0.036 7897	0.023 1632	0.014 6163	0.009 2433	0.003 7208	**96**
97	0.035 5456	0.022 2724	0.013 9868	0.008 8031	0.003 5102	**97**
98	0.034 3436	0.021 4157	0.013 3845	0.008 3840	0.003 3115	**98**
99	0.033 1822	0.020 5920	0.012 8082	0.007 9847	0.003 1241	**99**
100	0.032 0601	0.019 8000	0.012 2566	0.007 6045	0.002 9473	**100**

III. AMOUNT OF 1 PER ANNUM; $s_{\overline{n|}} = [(1+i)^n - 1]/i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	**1**
2	2.012 5000	2.015 0000	2.020 0000	2.025 0000	2.030 0000	**2**
3	3.037 6562	3.045 2250	3.060 4000	3.075 6250	3.090 9000	**3**
4	4.075 6269	4.090 9034	4.121 6080	4.152 5156	4.183 6270	**4**
5	5.126 5723	5.152 2669	5.204 0402	5.256 3285	5.309 1358	**5**
6	6.190 6544	6.229 5509	6.308 1210	6.387 7367	6.468 4099	**6**
7	7.268 0376	7.322 9942	7.434 2834	7.547 4301	7.662 4622	**7**
8	8.358 8881	8.432 8391	8.582 9691	8.736 1159	8.892 3360	**8**
9	9.463 3742	9.559 3317	9.754 6284	9.954 5188	10.159 1061	**9**
10	10.581 6664	10.702 7217	10.949 7210	11.203 3818	11.463 8793	**10**
11	11.713 9372	11.863 2625	12.168 7154	12.483 4663	12.807 7957	**11**
12	12.860 3614	13.041 2114	13.412 0897	13.795 5530	14.192 0296	**12**
13	14.021 1159	14.236 8296	14.680 3315	15.140 4418	15.617 7904	**13**
14	15.196 3799	15.450 3820	15.973 9382	16.518 9528	17.086 3242	**14**
15	16.386 3346	16.682 1378	17.293 4169	17.931 9267	18.598 9139	**15**
16	17.591 1638	17.932 3698	18.639 2853	19.380 2248	20.156 8813	**16**
17	18.811 0534	19.201 3554	20.012 0710	20.864 7304	21.761 5877	**17**
18	20.046 1915	20.489 3757	21.412 3124	22.386 3487	23.414 4354	**18**
19	21.296 7689	21.796 7164	22.840 5586	23.946 0074	25.116 8684	**19**
20	22.562 9785	23.123 6671	24.297 3698	25.544 6576	26.870 3745	**20**
21	23.845 0158	24.470 5221	25.783 3172	27.183 2741	28.676 4857	**21**
22	25.143 0785	25.837 5799	27.298 9835	28.862 8559	30.536 7803	**22**
23	26.457 3669	27.225 1436	28.844 9632	30.584 4273	32.452 8837	**23**
24	27.788 0840	28.633 5208	30.421 8625	32.349 0380	34.426 4702	**24**
25	29.135 4351	30.063 0236	32.030 2997	34.157 7639	36.459 2643	**25**
26	30.499 6280	31.513 9690	33.670 9057	36.011 7080	38.553 0423	**26**
27	31.880 8734	32.986 6785	35.344 3238	37.912 0007	40.709 6335	**27**
28	33.279 3843	34.481 4787	37.051 2103	39.859 8007	42.930 9225	**28**
29	34.695 3766	35.998 7009	38.792 2345	41.856 2958	45.218 8502	**29**
30	36.129 0688	37.538 6814	40.568 0792	43.902 7032	47.575 4157	**30**
31	37.580 6822	39.101 7616	42.379 4408	46.000 2707	50.002 6782	**31**
32	39.050 4407	40.688 2880	44.227 0296	48.150 2775	52.502 7585	**32**
33	40.538 5712	42.298 6123	46.111 5702	50.354 0344	55.077 8413	**33**
34	42.045 3033	43.933 0915	48.033 8016	52.612 8853	57.730 1765	**34**
35	43.570 8696	45.592 0879	49.994 4776	54.928 2074	60.462 0818	**35**
36	45.115 5055	47.275 9692	51.994 3672	57.301 4126	63.275 9443	**36**
37	46.679 4493	48.985 1087	54.034 2545	59.733 9479	66.174 2226	**37**
38	48.262 9424	50.719 8854	56.114 9396	62.227 2966	69.159 4493	**38**
39	49.866 2292	52.480 6837	58.237 2384	64.782 9791	72.234 2328	**39**
40	51.489 5571	54.267 8939	60.401 9832	67.402 5535	75.401 2597	**40**
41	53.133 1765	56.081 9123	62.610 0228	70.087 6174	78.663 2975	**41**
42	54.797 3412	57.923 1410	64.862 2233	72.839 8078	82.023 1965	**42**
43	56.482 3080	59.791 9881	67.159 4678	75.660 8030	85.483 8923	**43**
44	58.188 3369	61.688 8679	69.502 6571	78.552 3231	89.048 4091	**44**
46	59.915 6911	63.614 2010	71.892 7103	81.516 1312	92.719 8614	**45**
46	61.664 6372	65.568 4140	74.330 5645	84.554 0344	96.501 4572	**46**
47	63.435 4452	67.551 9402	76.817 1758	87.667 8853	100.396 5009	**47**
48	65.228 3884	69.565 2193	79.353 5193	90.859 5824	104.408 3960	**48**
49	67.043 7431	71.608 6976	81.940 5897	94.131 0720	108.540 6479	**49**
50	68.881 7899	73.682 8280	84.579 4015	97.484 3288	112.796 8673	**60**

III. AMOUNT OF 1 PER ANNUM; $s_{\overline{n}|} = [(1+i)^n - 1]/i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	70.742 8123	75.788 0705	87.270 9895	100.921 4575	117.180 7733	**51**
52	72.627 0974	77.924 8915	90.016 4093	104.444 4939	121.696 1965	**52**
53	74.534 9361	80.093 7649	92.816 7375	108.055 6063	126.347 0824	**53**
54	76.466 6228	82.295 1714	95.673 0722	111.756 9965	131.137 4949	**54**
55	78.422 4556	84.529 5989	98.586 5337	115.550 9214	136.071 6197	**55**
56	80.402 7363	86.797 5429	101.558 2643	119.439 6944	141.153 7683	**56**
57	82.407 7705	89.099 5061	104.589 4296	123.425 6868	146.388 3814	**57**
58	84.437 8676	91.435 9987	107.681 2182	127.511 3289	151.780 0328	**58**
59	86.493 3410	93.807 5386	110.834 8426	131.699 1121	157.333 4338	**59**
60	88.574 5078	96.214 6517	114.051 5394	135.991 5900	163.053 4368	**60**
61	90.681 6891	98.657 8715	117.332 5702	140.391 3797	168.945 0399	**61**
62	92.815 2102	101.137 7396	120.679 2216	144.901 1642	175.013 3911	**62**
63	94.975 4003	103.654 8057	124.092 8060	149.523 6933	181.263 7928	**63**
64	97.162 5928	106.209 6277	127.574 6622	154.261 7856	187.701 7066	**64**
65	99.377 1253	108.802 7722	131.126 1554	159.118 3303	194.332 7578	**65**
66	101.619 3393	111.434 8137	134.748 6785	164.096 2885	201.162 7406	**66**
67	103.889 5811	114.106 3359	138.443 6521	169.198 6957	208.197 6228	**67**
68	106.188 2008	116.817 9310	142.212 5251	174.428 6631	215.443 5515	**68**
69	108.515 5533	119.570 1999	146.056 7756	179.789 3797	222.906 8580	**69**
70	110.871 9978	122.363 7529	149.977 9111	185.284 1142	230.594 0637	**70**
71	113.257 8977	125.199 2092	153.977 4694	190.916 2171	238.511 8856	**71**
72	115.673 6215	128.077 1974	158.057 0188	196.689 1225	246.667 2422	**72**
73	118.119 5417	130.998 3553	162.218 1591	202.606 3506	255.067 2595	**73**
74	120.596 0360	133.963 3307	166.462 5223	208.671 5093	263.719 2773	**74**
75	123.103 4864	136.972 7806	170.791 7728	214.888 2970	272.630 8556	**75**
76	125.642 2800	140.027 3723	175.207 6082	221.260 5045	281.809 7813	**76**
77	128.212 8085	143.127 7829	179.711 7604	227.792 0171	291.264 0747	**77**
78	130.815 4686	146.274 6997	184.305 9956	234.486 8175	301.001 9969	**78**
79	133.450 6620	149.468 8202	188.992 1155	241.348 9879	311.032 0568	**79**
80	136.118 7953	152.710 8525	193.771 9578	248.382 7126	321.363 0185	**80**
81	138.820 2802	156.001 5153	198.647 3970	255.592 2805	332.003 9091	**81**
82	141.555 5337	159.341 5380	203.620 3449	262.982 0875	342.964 0264	**82**
83	144.324 9779	162.731 6611	208.692 7518	270.556 6397	354.252 9472	**83**
84	147.129 0401	166.172 6360	213.866 6068	278.320 5557	365.880 5356	**84**
85	149.968 1531	169.665 2255	219.143 9390	286.278 5695	377.856 9517	**85**
86	152.842 7550	173.210 2039	224.526 8177	294.435 5338	390.192 6602	**86**
87	155.753 2895	176.808 3569	230.017 3541	302.796 4221	402.898 4400	**87**
88	158.700 2056	180.460 4823	235.617 7012	311.366 3327	415.985 3932	**88**
89	161.683 9581	184.167 3895	241.330 0552	320.150 4910	429.464 9550	**89**
90	164.705 0076	187.929 9004	247.156 6563	329.154 2533	443.348 9037	**90**
91	167.763 8202	191.748 8489	253.099 7894	338.383 1096	457.649 3708	**91**
92	170.860 8680	195.625 0816	259.161 7852	347.842 6873	472.378 8519	**92**
93	173.996 6288	199.559 4578	265.345 0209	357.538 7545	487.550 2174	**93**
94	177.171 5867	203.552 8497	271.651 9214	367.477 2234	503.176 7240	**94**
95	180.386 2315	207.606 1425	278.084 9598	377.664 1540	519.272 0257	**95**
96	183.641 0594	211.720 2346	284.646 6590	388.105 7578	535.850 1865	**96**
97	186.936 5726	215.896 0381	291.339 5922	398.808 4018	552.925 6920	**97**
98	190.273 2798	220.134 4787	298.166 3840	409.778 6118	570.513 4628	**98**
99	193.651 6958	224.436 4959	305.129 7117	421.023 0771	588.628 8667	**99**
100	197.072 3420	228.803 0433	312.232 3059	432.548 6540	607.287 7327	**100**

III. AMOUNT OF 1 PER ANNUM; $s_{\overline{n}|} = [(1+i)^n - 1]/i$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	**1**
2	2.035 0000	2.040 0000	2.045 0000	2.050 0000	2.060 0000	**2**
3	3.106 2250	3.121 6000	3.137 0250	3.152 5000	3.183 6000	**3**
4	4.214 9429	4.246 4640	4.278 1911	4.310 1250	4.374 6160	**4**
5	5.362 4659	5.416 3226	5.470 7097	5.525 6312	5.637 0930	**5**
6	6.550 1522	6.632 9755	6.716 8917	6.801 9128	6.975 3185	**6**
7	7.779 4075	7.898 2945	8.019 1518	8.142 0085	8.393 8376	**7**
8	9.051 6868	9.214 2263	9.380 0136	9.549 1089	9.897 4679	**8**
9	10.368 4958	10.582 7953	10.802 1142	11.026 5643	11.491 3160	**9**
10	11.731 3932	12.006 1071	12.288 2094	12.577 8925	13.180 7949	**10**
11	13.141 9919	13.486 3514	13.841 1788	14.206 7872	14.971 6426	**11**
12	14.601 9616	15.025 8055	15.464 0318	15.917 1265	16.869 9412	**12**
13	16.113 0303	16.626 8377	17.159 9132	17.712 9828	18.882 1377	**13**
14	17.676 9864	18.291 9112	18.932 0194	19.598 6320	21.015 0659	**14**
15	19.295 6809	20.023 5876	20.784 0543	21.578 5636	23.275 9699	**15**
16	20.971 0297	21.824 5311	22.719 3367	23.657 4918	25.672 5281	**16**
17	22.705 0157	23.697 5124	24.741 7069	25.840 3664	28.212 8798	**17**
18	24.499 6913	25.645 4129	26.855 0837	28.132 3847	30.905 6525	**18**
19	26.357 1805	27.671 2294	29.063 5625	30.539 0039	33.759 9917	**19**
20	28.279 6818	29.778 0786	31.371 4228	33.065 9541	36.785 5912	**20**
21	30.269 4707	31.969 2017	33.783 1368	35.719 2518	39.992 7267	**21**
22	32.328 9021	34.247 9698	36.303 3780	38.505 2144	43.392 2903	**22**
23	34.460 4137	36.617 8886	38.937 0300	41.430 4751	46.995 8277	**23**
24	36.666 5282	39.082 6041	41.689 1963	44.501 9989	50.815 5774	**24**
25	38.949 8567	41.645 9083	44.565 2101	47.727 0988	54.864 5120	**25**
26	41.313 1017	44.311 7446	47.570 6446	51.113 4538	59.156 3827	**26**
27	43.759 0602	47.084 2144	50.711 3236	54.669 1264	63.705 7657	**27**
28	46.290 6273	49.967 5830	53.993 3332	58.402 5828	68.528 1116	**28**
29	48.910 7993	52.966 2863	57.423 0332	62.322 7119	73.639 7983	**29**
30	51.622 6773	56.084 9377	61.007 0697	66.438 8475	79.058 1862	**30**
31	54.429 4710	59.328 3353	64.752 3878	70.760 7899	84.801 6774	**31**
32	57.334 5025	62.701 4687	68.666 2452	75.298 8294	90.889 7780	**32**
33	60.341 2101	66.209 5274	72.756 2263	80.063 7708	97.343 1647	**33**
34	63.453 1524	69.857 9085	77.030 2565	85.066 9594	104.183 7546	**34**
36	66.674 0127	73.652 2249	81.496 6180	90.320 3074	111.434 7799	**35**
36	70.007 6032	77.598 3138	86.163 9658	95.836 3227	119.120 8667	**36**
37	73.457 8693	81.702 2464	91.041 3443	101.628 1389	127.268 1187	**37**
38	77.028 8947	85.970 3363	96.138 2048	107.709 5458	135.904 2058	**38**
39	80.724 9060	90.409 1497	101.464 4240	114.095 0231	145.058 4581	**39**
40	84.550 2777	95.025 5157	107.030 3231	120.799 7742	154.761 9656	**40**
41	88.509 5375	99.826 5363	112.846 6876	127.839 7630	165.047 6836	**41**
42	92.607 3713	104.819 5978	118.924 7885	135.231 7511	175.950 5446	**42**
43	96.848 6293	110.012 3817	125.276 4040	142.993 3387	187.507 5772	**43**
44	101.238 3313	115.412 8770	131.913 8422	151.143 0056	199.758 0319	**44**
45	105.781 6729	121.029 3920	138.849 9651	159.700 1559	212.743 5138	**45**
46	110.484 0314	126.870 5677	146.098 2135	168.685 1637	226.508 1246	**46**
47	115.350 9725	132.945 3904	153.672 6331	178.119 4218	241.098 6121	**47**
48	120.388 2566	139.263 2060	161.587 9016	188.025 3929	256.564 5288	**48**
49	125.601 8456	145.833 7343	169.859 3572	198.426 6626	272.958 4005	**49**
50	130.997 9102	152.667 0837	178.503 0283	209.347 9957	290.335 9046	**50**

III. AMOUNT OF 1 PER ANNUM; $s_{\overline{n}|} = [(1+i)^n - 1]/i$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	136.582 8370	159.773 7670	187.535 6646	220.815 3955	308.756 0589	**51**
52	142.363 2363	167.164 7177	196.974 7695	232.856 1653	328.281 4224	**52**
53	148.345 9496	174.851 3064	206.838 6341	245.498 9735	348.978 3077	**53**
54	154.538 0578	182.845 3586	217.146 3726	258.773 9222	370.917 0062	**54**
55	160.946 8898	191.159 1730	227.917 9594	272.712 6183	394.172 0266	**55**
56	166.580 0310	199.805 5399	239.174 2676	287.348 2492	418.882 3482	**56**
57	174.445 3321	208.797 7615	250.937 1096	302.715 6617	444.951 6890	**57**
58	181.550 9187	218.149 6720	263.229 2795	318.851 4448	472.648 7904	**58**
59	188.905 2008	227.875 6588	276.074 5971	335.794 0170	502.007 7178	**59**
60	196.516 8829	237.990 6852	289.497 9540	353.583 7179	533.128 1809	**60**
61	204.394 9738	248.510 3126	303.525 3619	372.262 9038	566.115 8717	**61**
62	212.548 7979	259.450 7251	318.184 0032	391.876 0490	601.082 8240	**62**
63	220.988 0058	270.828 7541	333.502 2833	412.469 8514	638.147 7935	**63**
64	229.722 5860	282.661 9043	349.509 8861	434.093 3440	677.436 6611	**64**
65	238.762 8765	294.968 3805	366.237 8310	456.798 0112	719.082 8608	**65**
66	248.119 5772	307.767 1157	383.718 5333	480.637 9117	763.227 8324	**66**
67	257.803 7624	321.077 8003	401.985 8673	505.669 8073	810.021 5024	**67**
68	267.826 8941	334.920 9123	421.075 2314	531.953 2977	859.622 7925	**68**
69	278.200 8354	349.317 7488	441.023 6168	559.550 9626	912.200 1600	**69**
70	288.937 8646	364.290 4588	461.869 6795	588.528 5107	967.932 1696	**70**
71	300.050 6899	379.862 0771	483.653 8151	618.954 9362	1027.008 0998	**71**
72	311.552 4640	396.056 5602	506.418 2368	650.902 6831	1089.628 5858	**72**
73	323.456 8002	412.898 8226	530.207 0575	684.447 8172	1156.006 3010	**73**
74	335.777 7882	430.414 7755	555.066 3751	719.670 2081	1226.366 6790	**74**
75	348.530 0108	448.631 3665	581.044 3619	756.653 7185	1300.948 6798	**75**
76	361.728 5612	467.576 6212	608.191 3582	795.486 4044	1380.005 6006	**76**
77	375.389 0609	487.279 6860	636.559 9694	836.260 7246	1463.805 9366	**77**
78	389.527 6780	507.770 8735	666.205 1680	879.073 7608	1552.634 2928	**78**
79	404.161 1467	529.081 7084	697.184 4005	924.027 4489	1646.792 3503	**79**
80	419.306 7868	551.244 9767	729.557 6985	971.228 8213	1746.599 8914	**80**
81	434.982 5244	574.294 7758	763.387 7950	1020.790 2624	1852.395 8849	**81**
82	451.206 9127	598.266 5668	798.740 2457	1072.829 7755	1964.539 6379	**82**
83	467.999 1547	623.197 2295	835.683 5568	1127.471 2643	2083.412 0162	**83**
84	485.379 1251	649.125 1187	874.289 3169	1184.844 8275	2209.416 7372	**84**
85	503.367 3945	676.090 1235	914.632 3361	1245.087 0689	2342.981 7414	**85**
86	521.985 2533	704.133 7284	956.790 7912	1308.341 4223	2484.560 6459	**86**
87	541.254 7372	733.299 0775	1000.846 3769	1374.758 4935	2634.634 2847	**87**
88	561.198 6530	763.631 0406	1046.884 4638	1444.496 4181	2793.712 3417	**88**
89	581.840 6058	795.176 2823	1094.994 2647	1517.721 2390	2962.335 0822	**89**
90	603.205 0270	827.983 3335	1145.269 0066	1594.607 3010	3141.075 1872	**90**
91	625.317 2030	862.102 6669	1197.806 1119	1675.337 6660	3330.539 6984	**91**
92	648.203 3051	897.586 7736	1252.707 3869	1760.104 5493	3531.372 0803	**92**
93	671.890 4207	934.490 2445	1310.079 2193	1849.109 7768	3744.254 4051	**93**
94	696.406 5855	972.869 8543	1370.032 7842	1942.565 2656	3969.909 6694	**94**
95	721.780 8160	1012.784 6485	1432.684 2595	2040.693 5289	4209.104 2496	**95**
96	748.043 1445	1054.296 0344	1498.155 0512	2143.728 2054	4462.650 5046	**96**
97	775.224 6546	1097.467 8758	1566.572 0285	2251.914 6156	4731.409 5348	**97**
98	803.357 5175	1142.366 5908	1638.067 7698	2365.510 3464	5016.294 1070	**98**
99	832.475 0306	1189.061 2544	1712.780 8194	2484.785 8637	5318.271 7534	**99**
100	862.611 6567	1237.623 7046	1790.855 9563	2610.025 1569	5638.368 0586	**100**

IV. PRESENT VALUE OF 1 PER ANNUM; $a_{\overline{n}|} = (1 - v^n)/i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	**1**
2	1.963 1154	1.955 8834	1.941 5609	1.927 4242	1.913 4697	**2**
3	2.926 5307	2.912 2004	2.883 8833	2.856 0236	2.828 6114	**3**
4	3.878 0580	3.854 3846	3.807 7287	3.761 9742	3.717 0984	**4**
5	4.817 8350	4.782 6450	4.713 4595	4.645 8285	4.579 7072	**5**
6	5.746 0099	5.697 1872	5.601 4309	5.508 1254	5.417 1914	**6**
7	6.662 7258	6.598 2140	6.471 9911	6.349 3906	6.230 2830	**7**
8	7.568 1243	7.485 9251	7.325 4814	7.170 1372	7.019 6922	**8**
9	8.462 3450	8.360 5173	8.162 2367	7.970 8655	7.786 1089	**9**
10	9.345 5259	9.222 1846	8.982 5850	8.752 0639	8.530 2028	**10**
11	10.217 8034	10.071 1178	9.786 8480	9.514 2087	9.252 6241	**11**
12	11.079 3120	10.907 5052	10.575 3412	10.257 7646	9.954 0040	**12**
13	11.930 1847	11.731 5322	11.348 3737	10.983 1850	10.634 9553	**13**
14	12.770 5527	12.543 3815	12.106 2488	11.690 9122	11.296 0731	**14**
15	13.600 5459	13.343 2330	12.849 2635	12.381 3777	11.937 9351	**15**
16	14.420 2923	14.131 2641	13.577 7093	13.055 0027	12.561 1020	**16**
17	15.229 9183	14.907 6493	14.291 8719	13.712 1977	13.166 1185	**17**
18	16.029 5489	15.672 5609	14.992 0312	14.353 3636	13.753 5131	**18**
19	16.819 3076	16.426 1684	15.678 4620	14.978 8913	14.323 7991	**19**
20	17.599 3161	17.168 6388	16.351 4333	15.589 1623	14.877 4749	**20**
21	18.369 6949	17.900 1367	17.011 2092	16.184 5486	15.415 0241	**21**
22	19.130 5629	18.620 8244	17.658 0482	16.765 4132	15.936 9166	**22**
23	19.882 0374	19.330 8614	18.292 2041	17.332 1105	16.443 6084	**23**
24	20.624 2345	20.030 4054	18.913 9256	17.884 9858	16.935 5421	**24**
25	21.357 2686	20.719 6112	19.523 4565	18.424 3764	17.413 1477	**25**
26	22.081 2530	21.398 6317	20.121 0358	18.950 6111	17.876 8424	**26**
27	22.796 2993	22.067 6175	20.706 8978	19.464 0109	18.327 0315	**27**
28	23.502 5178	22.726 7167	21.281 2724	19.964 8887	18.764 1082	**28**
29	24.200 0176	23.376 0756	21.844 3847	20.453 5499	19.188 4546	**29**
30	24.888 9062	24.015 8380	22.396 4556	20.930 2926	19.600 4413	**30**
31	25.569 2901	24.646 1458	22.937 7015	21.395 4074	20.000 4285	**31**
32	26.241 2742	25.267 1387	23.468 3348	21.849 1780	20.388 7655	**32**
33	26.904 9621	25.878 9544	23.988 5636	22.291 8809	20.765 7918	**33**
34	27.560 4564	26.481 7285	24.498 5917	22.723 7863	21.131 8367	**34**
35	28.207 8582	27.075 5946	24.998 6193	23.145 1573	21.487 2201	**35**
36	28.847 2674	27.660 6843	25.488 8425	23.556 2511	21.832 2525	**36**
37	29.478 7826	28.237 1274	25.969 4534	23.957 3181	22.167 2354	**37**
38	30.102 5013	28.805 0516	26.440 6406	24.348 6030	22.492 4616	**38**
39	30.718 5198	29.364 5829	26.902 5888	24.730 3444	22.808 2151	**39**
40	31.326 9332	29.915 8452	27.355 4792	25.102 7750	23.114 7720	**40**
41	31.927 8352	30.458 9608	27.799 4895	25.466 1220	23.412 4000	**41**
42	32.521 3187	30.994 0500	28.234 7936	25.820 6068	23.701 3592	**42**
43	33.107 4753	31.521 2316	28.661 5623	26.166 4457	23.981 9021	**43**
44	33.686 3954	32.040 6222	29.079 9631	26.503 8495	24.254 2739	**44**
45	34.258 1682	32.552 3372	29.490 1599	26.833 0239	24.518 7125	**45**
46	34.822 8822	33.056 4898	29.892 3136	27.154 1696	24.775 4491	**46**
47	35.380 6244	33.553 1919	30.286 5820	27.467 4826	25.024 7078	**47**
48	35.931 4809	34.042 5536	30.673 1196	27.773 1537	25.266 7066	**48**
49	36.475 5367	34.524 6834	31.052 0780	28.071 3695	25.501 6569	**49**
50	37.012 8757	34.999 6881	31.423 6059	28.362 3117	25.729 7640	**50**

IV. PRESENT VALUE OF 1 PER ANNUM; $a_{\overline{n|}} = (1 - v^n)/i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	37.543 5810	35.467 6730	31.787 8489	28.646 1577	25.951 2272	**51**
52	38.067 7343	35.928 7418	32.144 9499	28.923 0807	26.166 2400	**52**
53	38.585 4166	36.382 9969	32.495 0489	29.193 2495	26.374 9903	**53**
54	39.096 7078	36.830 5388	32.838 2833	29.456 8288	26.577 6605	**54**
55	39.601 6867	37.271 4668	33.174 7875	29.713 9793	26.774 4276	**55**
56	40.100 4313	37.705 8786	33.504 6936	29.964 8578	26.965 4637	**56**
57	40.593 0185	38.133 8706	33.828 1310	30.209 6174	27.150 9357	**57**
58	41.079 5245	38.555 5375	34.145 2265	30.448 4072	27.331 0055	**58**
59	41.560 0242	38.970 9729	34.456 1044	30.681 3729	27.505 8306	**59**
60	42.034 5918	39.380 2689	34.760 8867	30.908 6565	27.675 5637	**60**
61	42.503 3005	39.783 5161	35.059 6928	31.130 3966	27.840 3531	**61**
62	42.966 2227	40.180 8041	35.352 6400	31.346 7284	28.000 3428	**62**
63	43.423 4299	40.572 2208	35.639 8432	31.557 7838	28.155 6726	**63**
64	43.874 9925	40.957 8530	35.921 4149	31.763 6915	28.306 4783	**64**
65	44.320 9802	41.337 7862	36.197 4655	31.964 5771	28.452 8915	**65**
66	44.761 4619	41.712 1046	36.468 1035	32.160 5630	28.595 0403	**66**
67	45.196 5056	42.080 8912	36.733 4348	32.351 7688	28.733 0488	**67**
68	45.626 1784	42.444 2278	36.993 5635	32.538 3110	28.867 0377	**68**
69	46.050 5466	42.802 1949	37.248 5917	32.720 3034	28.997 1240	**69**
70	46.469 6756	43.154 8718	37.498 6193	32.897 8570	29.123 4214	**70**
71	46.883 6302	43.502 3368	37.743 7444	33.071 0800	29.246 0401	**71**
72	47.292 4743	43.844 6668	37.984 0631	33.240 0780	29.365 0875	**72**
73	47.696 2709	44.181 9577	38.219 6697	33.404 9542	29.480 6675	**73**
74	48.095 0824	44.514 2243	38.450 6566	33.565 8089	29.592 8811	**74**
75	48.488 9703	44.841 6003	38.677 1143	33.722 7404	29.701 8263	**75**
76	48.877 9953	45.164 1383	38.899 1317	33.875 8443	29.807 5983	**76**
77	49.262 2176	45.481 9096	39.116 7958	34.025 2140	29.910 2896	**77**
78	49.641 6964	45.794 9848	39.330 1919	34.170 9405	30.009 9899	**78**
79	50.016 4903	46.103 4333	39.539 4039	34.313 1127	30.106 7863	**79**
80	50.386 6571	46.407 3235	39.744 5136	34.451 8172	30.200 7634	**80**
81	50.752 2539	46.706 7226	39.945 6016	34.587 1388	30.292 0033	**81**
82	51.113 3372	47.001 6972	40.142 7466	34.719 1598	30.380 5858	**82**
83	51.469 9626	47.292 3125	40.336 0261	34.847 9607	30.466 5881	**83**
84	51.822 1853	47.578 6330	40.525 5158	34.973 6202	30.550 0856	**84**
85	52.170 0596	47.860 7222	40.711 2900	35.096 2149	30.631 1510	**85**
86	52.513 6391	48.138 6425	40.893 4216	35.215 8194	30.709 8554	**86**
87	52.852 9769	48.412 4557	41.071 9819	35.332 5067	30.786 2673	**87**
88	53.188 1253	48.682 2224	41.247 0411	35.446 3480	30.860 4537	**88**
89	53.519 1361	48.948 0023	41.418 6677	35.557 4127	30.932 4794	**89**
90	53.846 0603	49.209 8545	41.586 9292	35.665 7685	31.002 4071	**90**
91	54.168 9485	49.467 8370	41.751 8913	35.771 4814	31.070 2982	**91**
92	54.487 8504	49.722 0069	41.913 6190	35.874 6160	31.136 2118	**92**
93	54.802 8152	49.972 4205	42.072 1754	35.975 2352	31.200 2057	**93**
94	55.113 8915	50.219 1335	42.227 6230	36.073 4002	31.262 3356	**94**
95	55.421 1274	50.462 2005	42.380 0225	36.169 1709	31.322 6559	**95**
96	55.724 5703	50.701 6754	42.529 4339	36.262 6057	31.381 2193	**96**
97	56.024 2670	50.937 6112	42.675 9155	36.353 7617	31.438 0770	**97**
98	56.320 2637	51.170 0603	42.819 5250	36.442 6943	31.493 2787	**98**
99	56.612 6061	51.399 0742	42.960 3187	36.529 4579	31.546 8725	**99**
100	56.901 3394	51.624 7037	43.098 3516	36.614 1053	31.598 9053	**100**

IV. PRESENT VALUE OF 1 PER ANNUM; $a_{\overline{n}|} = (1 - v^n)/i$

n	**$3\frac{1}{2}$%**	**4%**	**$4\frac{1}{2}$%**	**5%**	**6%**	**n**
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	**1**
2	1.899 6943	1.886 0947	1.872 6678	1.859 4104	1.833 3927	**2**
3	2.801 6370	2.775 0910	2.748 9644	2.723 2480	2.673 0119	**3**
4	3.673 0792	3.629 8952	3.587 5257	3.545 9505	3.465 1056	**4**
5	4.515 0524	4.451 8223	4.389 9767	4.329 4767	4.212 3638	**5**
6	5.328 5530	5.242 1369	5.157 8725	5.075 6921	4.917 3243	**6**
7	6.114 5440	6.002 0547	5.892 7009	5.786 3734	5.582 3814	**7**
8	6.873 9555	6.732 7449	6.595 8861	6.463 2128	6.209 7938	**8**
9	7.607 6865	7.435 3316	7.268 7905	7.107 8217	6.801 6923	**9**
10	8.316 6053	8.110 8958	7.912 7182	7.721 7349	7.360 0871	**10**
11	9.001 5510	8.760 4767	8.528 9169	8.306 4142	7.886 8746	**11**
12	9.663 3343	9.385 0738	9.118 5808	8.863 2516	8.383 8439	**12**
13	10.302 7385	9.985 6478	9.682 8524	9.393 5730	8.852 6830	**13**
14	10.920 5203	10.563 1229	10.222 8253	9.898 6409	9.294 9839	**14**
15	11.517 4109	11.118 3874	10.739 5457	10.379 6580	9.712 2490	**15**
16	12.094 1168	11.652 2956	11.234 0150	10.837 7696	10.105 8953	**16**
17	12.651 3206	12.165 6689	11.707 1914	11.274 0662	10.477 2597	**17**
18	13.189 6817	12.659 2970	12.159 9918	11.689 5869	10.827 6035	**18**
19	13.709 8374	13.133 9394	12.593 2936	12.085 3209	11.158 1165	**19**
20	14.212 4033	13.590 3263	13.007 9365	12.462 2103	11.469 9212	**20**
21	14.697 9742	14.029 1599	13.404 7239	12.821 1527	11.764 0766	**21**
22	15.167 1248	14.451 1153	13.784 4248	13.163 0026	12.041 5817	**22**
23	15.620 4105	14.856 8417	14.147 7749	13.488 5739	12.303 3790	**23**
24	16.058 3676	15.246 9631	14.495 4784	13.798 6418	12.550 3575	**24**
25	16.481 5146	15.622 0799	14.828 2090	14.093 9446	12.783 3562	**25**
26	16.890 3523	15.982 7692	15.146 6114	14.375 1853	13.003 1662	**26**
27	17.285 3645	16.329 5857	15.451 3028	14.643 0336	13.210 5341	**27**
28	17.667 0188	16.663 0632	15.742 8735	14.898 1273	13.406 1643	**28**
29	18.035 7670	16.983 7146	16.021 8885	15.141 0736	13.590 7210	**29**
30	18.392 0454	17.292 0333	16.288 8885	15.372 4510	13.764 8312	**30**
31	18.736 2758	17.588 4936	16.544 3910	15.592 8105	13.929 0860	**31**
32	19.068 8655	17.873 5515	16.788 8909	15.802 6767	14.084 0434	**32**
33	19.390 2082	18.147 6457	17.022 8621	16.002 5492	14.230 2296	**33**
34	19.700 6842	18.411 1978	17.246 7580	16.192 9040	14.368 1411	**34**
35	20.000 6611	18.664 6132	17.461 0124	16.374 1943	14.498 2464	**35**
36	20.290 4938	18.908 2820	17.666 0406	16.546 8517	14.620 9871	**36**
37	20.570 5254	19.142 5788	17.862 2398	16.711 2873	14.736 7803	**37**
38	20.841 0874	19.367 8642	18.049 9902	16.867 8927	14.846 0192	**38**
39	21.102 4999	19.584 4848	18.229 6557	17.017 0407	14.949 0747	**39**
40	21.355 0723	19.792 7739	18.401 5844	17.159 0864	15.046 2969	**40**
41	21.599 1037	19.993 0518	18.566 1095	17.294 3680	15.138 0159	**41**
42	21.834 8828	20.185 6267	18.723 5498	17.423 2076	15.224 5433	**42**
43	22.062 6887	20.370 7949	18.874 2103	17.545 9120	15.306 1729	**43**
44	22.282 7910	20.548 8413	19.018 3831	17.662 7733	15.383 1820	**44**
45	22.495 4503	20.720 0397	19.156 3474	17.774 0698	15.455 8321	**45**
46	22.700 9181	20.884 6536	19.288 3707	17.880 0665	15.524 3699	**46**
47	22.899 4378	21.042 9361	19.414 7088	17.981 0157	15.589 0282	**47**
48	23.091 2443	21.195 1309	19.535 6065	18.077 1578	15.650 0266	**48**
49	23.276 5645	21.341 4720	19.651 2981	18.168 7217	15.707 5723	**49**
50	23.455 6179	21.482 1846	19.762 0078	18.255 9255	15.761 8606	**50**

IV. PRESENT VALUE OF 1 PER ANNUM; $a_{\overline{n}|} = (1 - v^n)/i$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	23.628 6163	21.617 4852	19.867 9500	18.338 9766	15.813 0761	**51**
52	23.795 7645	21.747 5819	19.969 3302	18.418 0730	15.861 3925	**52**
53	23.957 2604	21.872 6749	20.066 3447	18.493 4028	15.906 9741	**53**
54	24.113 2951	21.992 9567	20.159 1815	18.565 1456	15.949 9755	**54**
55	24.264 0532	22.108 6122	20.248 0206	18.633 4720	15.990 5430	**55**
56	24.409 7133	22.219 8194	20.333 0340	18.698 5447	16.028 8141	**56**
57	24.550 4776	22.326 7494	20.414 3866	18.760 5188	16.064 9190	**57**
58	24.686 4228	22.429 5668	20.492 2360	18.819 5417	16.098 9802	**58**
59	24.817 7998	22.528 4296	20.566 7330	18.875 7540	16.131 1134	**59**
60	24.944 7341	22.623 4900	20.638 0220	18.929 2895	16.161 4277	**60**
61	25.067 3760	22.714 8942	20.706 2412	18.980 2757	16.190 0261	**61**
62	25.185 8705	22.802 7829	20.771 5227	19.028 8340	16.217 0058	**62**
63	25.300 3580	22.887 2912	20.833 9930	19.075 0800	16.242 4583	**63**
64	25.410 9739	22.968 5493	20.893 7732	19.119 1238	16.266 4701	**64**
65	25.517 8492	23.046 6820	20.950 9791	19.161 0703	16.289 1227	**65**
66	25.621 1103	23.121 8096	21.005 7217	19.201 0194	16.310 4931	**66**
67	25.720 8795	23.194 0477	21.058 1068	19.239 0661	16.330 6539	**67**
68	25.817 2749	23.263 5074	21.108 2362	19.275 3010	16.349 6735	**68**
69	25.910 4105	23.330 2956	21.156 2069	19.309 8105	16.367 6165	**69**
70	26.000 3966	23.394 5150	21.202 1119	19.342 6766	16.384 5439	**70**
71	26.087 3398	23.456 2644	21.246 0401	19.373 9778	16.400 5131	**71**
72	26.171 3428	23.515 6388	21.288 0766	19.403 7883	16.415 5784	**72**
73	26.252 5051	23.572 7297	21.328 3030	19.432 1794	16.429 7909	**73**
74	26.330 9228	23.627 6247	21.366 7971	19.459 2185	16.443 1990	**74**
75	26.406 6887	23.680 4083	21.403 6336	19.484 9700	16.455 8481	**75**
76	26.479 8924	23.731 1619	21.438 8838	19.509 4952	16.467 7812	**76**
77	26.550 6207	23.779 9633	21.472 6161	19.532 8526	16.479 0389	**77**
78	26.618 9572	23.826 8878	21.504 8958	19.555 0977	16.489 6593	**78**
79	26.684 9828	23.872 0075	21.535 7854	19.576 2835	16.499 6786	**79**
80	26.748 7757	23.915 3918	21.565 3449	19.596 4605	16.509 1308	**80**
81	26.810 4113	23.957 1075	21.593 6315	19.615 6767	16.518 0479	**81**
82	26.869 9626	23.997 2188	21.620 7000	19.633 9778	16.526 4603	**82**
83	26.927 5001	24.035 7873	21.646 6029	19.651 4074	16.534 3965	**83**
84	26.983 0919	24.072 8724	21.671 3903	19.668 0070	16.541 8835	**84**
85	27.036 8037	24.108 5312	21.695 1103	19.683 8162	16.548 9467	**85**
86	27.088 6993	24.142 8184	21.717 8089	19.698 8726	16.555 6101	**86**
87	27.138 8399	24.175 7869	21.739 5301	19.713 2120	16.561 8963	**87**
88	27.187 2849	24.207 4874	21.760 3159	19.726 8686	16.567 8267	**88**
89	27.234 0917	24.237 9687	21.780 2066	19.739 8748	16.573 4214	**89**
90	27.279 3156	24.267 2776	21.799 2407	19.752 2617	16.578 6994	**90**
91	27.323 0103	24.295 4592	21.817 4553	19.764 0588	16.583 6787	**91**
92	27.365 2273	24.322 5569	21.834 8854	19.775 2941	16.588 3762	**92**
93	27.406 0167	24.348 6124	21.851 5650	19.785 9944	16.592 8077	**93**
94	27.445 4268	24.373 6658	21.867 5263	19.796 1851	16.596 9884	**94**
95	27.483 5042	24.397 7556	21.882 8003	19.805 8906	16.600 9324	**95**
96	27.520 2939	24.420 9188	21.897 4166	19.815 1339	16.604 6532	**96**
97	27.555 8395	24.443 1912	21.911 4034	19.823 9370	16.608 1634	**97**
98	27.590 1831	24.464 6069	21.924 7879	19.832 3210	16.611 4749	**98**
99	27.623 3653	24.485 1990	21.937 5961	19.840 3057	16.614 5990	**99**
100	27.655 4254	24.504 9990	21.949 8527	19.847 9102	16.617 5462	**100**

V. ANNUITY THAT 1 WILL PURCHASE; $1/a_{\overline{n}|} = i/(1-v^n)$; $1/s_{\overline{n}|} = 1/a_{\overline{n}|} - i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	**1**
2	0.509 3944	0.511 2779	0.515 0495	0.518 8272	0.522 6108	**2**
3	0.341 7012	0.343 3830	0.346 7547	0.350 1372	0.353 5304	**3**
4	0.257 8610	0.259 4448	0.262 6238	0.265 8179	0.269 0270	**4**
5	0.207 5621	0.209 0893	0.212 1584	0.215 2469	0.218 3546	**5**
6	0.174 0338	0.175 5252	0.178 5258	0.181 5500	0.184 5975	**6**
7	0.150 0887	0.151 5562	0.154 5120	0.157 4954	0.160 5064	**7**
8	0.132 1331	0.133 5840	0.136 5098	0.139 4673	0.142 4564	**8**
9	0.118 1705	0.119 6098	0.122 5154	0.125 4569	0.128 4339	**9**
10	0.107 0031	0.108 4342	0.111 3265	0.114 2588	0.117 2305	**10**
11	0.097 8684	0.099 2938	0.102 1779	0.105 1060	0.108 0774	**11**
12	0.090 2583	0.091 6800	0.094 5596	0.097 4871	0.100 4621	**12**
13	0.083 8210	0.085 2404	0.088 1184	0.091 0483	0.094 0295	**13**
14	0.078 3051	0.079 7233	0.082 6020	0.085 5365	0.088 5263	**14**
15	0.073 5265	0.074 9444	0.077 8255	0.080 7665	0.083 7666	**15**
16	0.069 3467	0.070 7651	0.073 6501	0.076 5990	0.079 6108	**16**
17	0.065 6602	0.067 0797	0.069 9698	0.072 9278	0.075 9525	**17**
18	0.062 3848	0.063 8058	0.066 7021	0.069 6701	0.072 7087	**18**
19	0.059 4555	0.060 8785	0.063 7818	0.066 7606	0.069 8139	**19**
20	0.056 8204	0.058 2457	0.061 1567	0.064 1471	0.067 2157	**20**
21	0.054 4375	0.055 8655	0.058 7847	0.061 7873	0.064 8718	**21**
22	0.052 2724	0.053 7033	0.056 6314	0.059 6466	0.062 7474	**22**
23	0.050 2967	0.051 7308	0.054 6681	0.057 6964	0.060 8139	**23**
24	0.048 4866	0.049 9241	0.052 8711	0.055 9128	0.059 0474	**24**
25	0.046 8225	0.048 2635	0.051 2204	0.054 2759	0.057 4279	**25**
26	0.045 2873	0.046 7320	0.049 6992	0.052 7687	0.055 9383	**26**
27	0.043 8668	0.045 3153	0.048 2931	0.051 3769	0.054 5642	**27**
28	0.042 5486	0.044 0011	0.046 9897	0.050 0879	0.053 2932	**28**
29	0.041 3223	0.042 7788	0.045 7784	0.048 8913	0.052 1147	**29**
30	0.040 1785	0.041 6392	0.044 6499	0.047 7776	0.051 0193	**30**
31	0.039 1094	0.040 5743	0.043 5963	0.046 7390	0.049 9989	**31**
32	0.038 1079	0.039 5771	0.042 6106	0.045 7683	0.049 0466	**32**
33	0.037 1679	0.038 6414	0.041 6865	0.044 8594	0.048 1561	**33**
34	0.036 2839	0.037 7619	0.040 8187	0.044 0067	0.047 3220	**34**
35	0.035 4511	0.036 9336	0.040 0022	0.043 2056	0.046 5393	**35**
36	0.034 6653	0.036 1524	0.039 2329	0.042 4516	0.045 8038	**36**
37	0.033 9227	0.035 4144	0.038 5068	0.041 7409	0.045 1116	**37**
38	0.033 2198	0.034 7161	0.037 8206	0.041 0701	0.044 4593	**38**
39	0.032 5537	0.034 0546	0.037 1711	0.040 4362	0.043 8439	**39**
40	0.031 9214	0.033 4271	0.036 5557	0.039 8362	0.043 2624	**40**
41	0.031 3206	0.032 8311	0.035 9719	0.039 2679	0.042 7124	**41**
42	0.030 7491	0.032 2643	0.035 4173	0.038 7288	0.042 1917	**42**
43	0.030 2047	0.031 7246	0.034 8899	0.038 2169	0.041 6981	**43**
44	0.029 6856	0.031 2104	0.034 3879	0.037 7304	0.041 2298	**44**
45	0.029 1901	0.030 7198	0.033 9096	0.037 2675	0.040 7852	**45**
46	0.028 7167	0.030 2512	0.033 4534	0.036 8268	0.040 3625	**46**
47	0.028 2641	0.029 8034	0.033 0179	0.036 4067	0.039 9605	**47**
48	0.027 8307	0.029 3750	0.032 6018	0.036 0060	0.039 5778	**48**
49	0.027 4156	0.028 9648	0.032 2040	0.035 6235	0.039 2131	**49**
50	0.027 0176	0.028 5717	0.031 8232	0.035 2581	0.038 8655	**50**

V. ANNUITY THAT 1 WILL PURCHASE; $1/a_{\overline{n}|} = i/(1-v^n)$; $1/s_{\overline{n}|} = 1/a_{\overline{n}|} - i$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
51	0.026 6357	0.028 1947	0.031 4586	0.034 9087	0.038 5338	**51**
52	0.026 2690	0.027 8329	0.031 1091	0.034 5745	0.038 2172	**52**
53	0.025 9165	0.027 4854	0.030 7739	0.034 2545	0.037 9147	**53**
54	0.025 5776	0.027 1514	0.030 4523	0.033 9480	0.037 6256	**54**
55	0.025 2514	0.026 8302	0.030 1434	0.033 6542	0.037 3491	**55**
56	0.024 9374	0.026 5211	0.029 8466	0.033 3724	0.037 0845	**56**
57	0.024 6348	0.026 2234	0.029 5612	0.033 1020	0.036 8311	**57**
58	0.024 3430	0.025 9366	0.029 2867	0.032 8424	0.036 5885	**58**
59	0.024 0616	0.025 6601	0.029 0224	0.032 5931	0.036 3559	**59**
60	0.023 7899	0.025 3934	0.028 7680	0.032 3534	0.036 1330	**60**
61	0.023 5276	0.025 1360	0.028 5228	0.032 1229	0.035 9191	**61**
62	0.023 2741	0.024 8875	0.028 2864	0.031 9013	0.035 7139	**62**
63	0.023 0290	0.024 6474	0.028 0585	0.031 6879	0.035 5168	**63**
64	0.022 7920	0.024 4153	0.027 8385	0.031 4825	0.035 3276	**64**
65	0.022 5627	0.024 1909	0.027 6262	0.031 2846	0.035 1458	**65**
66	0.022 3406	0.023 9739	0.027 4212	0.031 0940	0.034 9711	**66**
67	0.022 1256	0.023 7638	0.027 2232	0.030 9102	0.034 8031	**67**
68	0.021 9172	0.023 5603	0.027 0317	0.030 7330	0.034 6416	**68**
69	0.021 7153	0.023 3633	0.026 8467	0.030 5621	0.034 4862	**69**
70	0.021 5194	0.023 1724	0.026 6676	0.030 3971	0.034 3366	**70**
71	0.021 3294	0.022 9873	0.026 4945	0.030 2379	0.034 1927	**71**
72	0.021 1450	0.022 8078	0.026 3268	0.030 0842	0.034 0540	**72**
73	0.020 9660	0.022 6337	0.026 1645	0.029 9357	0.033 9205	**73**
74	0.020 7921	0.022 4647	0.026 0074	0.029 7922	0.033 7919	**74**
75	0.020 6232	0.022 3007	0.025 8551	0.029 6536	0.033 6680	**75**
76	0.020 4591	0.022 1415	0.025 7075	0.029 5196	0.033 5485	**76**
77	0.020 2995	0.021 9868	0.025 5645	0.029 3900	0.033 4333	**77**
78	0.020 1444	0.021 8365	0.025 4258	0.029 2646	0.033 3222	**78**
79	0.019 9934	0.021 6904	0.025 2912	0.029 1434	0.033 2151	**79**
80	0.019 8465	0.021 5483	0.025 1607	0.029 0260	0.033 1117	**80**
81	0.019 7036	0.021 4102	0.025 0340	0.028 9125	0.033 0120	**81**
82	0.019 5644	0.021 2758	0.024 9111	0.028 8025	0.032 9158	**82**
83	0.019 4288	0.021 1451	0.124 7917	0.028 6961	0.032 8228	**83**
84	0.019 2968	0.021 0178	0.024 6758	0.028 5930	0.032 7331	**84**
85	0.019 1681	0.020 8940	0.024 5632	0.028 4931	0.032 6465	**85**
86	0.019 0427	0.020 7733	0.024 4538	0.028 3963	0.032 5628	**86**
87	0.018 9204	0.020 6558	0.024 3475	0.028 3025	0.032 4820	**87**
88	0.018 8012	0.020 5414	0.024 2442	0.028 2116	0.032 4039	**88**
89	0.018 6849	0.020 4298	0.024 1437	0.028 1235	0.032 3285	**89**
90	0.018 5715	0.020 3211	0.024 0460	0.028 0381	0.032 2556	**90**
91	0.018 4608	0.020 2152	0.023 9510	0.027 9552	0.032 1851	**91**
92	0.018 3527	0.020 1118	0.023 8586	0.027 8749	0.032 1169	**92**
93	0.018 2472	0.020 0110	0.023 7687	0.027 7969	0.032 0511	**93**
94	0.018 1442	0.019 9127	0.023 6812	0.027 7213	0.031 9874	**94**
95	0.018 0437	0.019 8168	0.023 5960	0.027 6479	0.031 9258	**95**
96	0.017 9454	0.019 7232	0.023 5131	0.027 5766	0.031 8662	**96**
97	0.017 8494	0.019 6319	0.023 4324	0.027 5075	0.031 8086	**97**
98	0.017 7556	0.019 5427	0.023 3538	0.027 4403	0.031 7528	**98**
99	0.017 6639	0.019 4556	0.023 2773	0.027 3752	0.031 6989	**99**
100	0.017 5743	0.019 3706	0.023 2027	0.027 3119	0.031 6467	**100**

V. ANNUITY THAT 1 WILL PURCHASE; $1/a_{\overline{n}|}=i/(1-v^n)$; $1/s_{\overline{n}|}=1/a_{\overline{n}|}-i$

n	$3\frac{1}{2}$%	4%	$4\frac{1}{2}$%	5%	6%	**n**
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	**1**
2	0.526 4005	0.530 1961	0.533 9976	0.537 8049	0.545 4369	**2**
3	0.356 9342	0.360 3485	0.363 7734	0.367 2086	0.374 1098	**3**
4	0.272 2511	0.275 4900	0.278 7436	0.282 0118	0.288 5915	**4**
5	0.221 4814	0.224 6271	0.227 7916	0.230 9748	0.237 3964	**5**
6	0.187 6682	0.190 7619	0.193 8784	0.197 0175	0.203 3626	**6**
7	0.163 5445	0.166 6096	0.169 7015	0.172 8198	0.179 1350	**7**
8	0.145 4766	0.148 5278	0.151 6097	0.154 7218	0.161 0359	**8**
9	0.131 4460	0.134 4930	0.137 5745	0.140 6901	0.147 0222	**9**
10	0.120 2414	0.123 2909	0.126 3788	0.129 5046	0.135 8680	**10**
11	0.111 0920	0.114 1490	0.117 2482	0.120 3889	0.126 7929	**11**
12	0.103 4839	0.106 5522	0.109 6662	0.112 8254	0.119 2770	**12**
13	0.097 0616	0.100 1437	0.103 2754	0.106 4558	0.112 9601	**13**
14	0.091 5707	0.094 6690	0.097 8203	0.101 0240	0.107 5849	**14**
15	0.086 8251	0.089 9411	0.093 1138	0.096 3423	0.102 9628	**16**
16	0.082 6848	0.085 8200	0.089 0154	0.092 2699	0.098 9521	**16**
17	0.079 0431	0.082 1985	0.085 4176	0.088 6991	0.095 4448	**17**
18	0.075 8168	0.078 9933	0.082 2369	0.085 5462	0.092 3565	**18**
19	0.072 9403	0.076 1386	0.079 4073	0.082 7450	0.089 6209	**19**
20	0.070 3611	0.073 5817	0.076 8761	0.080 2426	0.087 1846	**20**
21	0.068 0366	0.071 2801	0.074 6006	0.077 9961	0.085 0046	**21**
22	0.065 9321	0.069 1988	0.072 5456	0.075 9705	0.083 0456	**22**
23	0.064 0188	0.067 3091	0.070 6825	0.074 1368	0.081 2785	**23**
24	0.062 2728	0.065 5868	0.068 9870	0.072 4709	0.079 6790	**24**
25	0.060 6740	0.064 0120	0.067 4390	0.070 9525	0.078 2267	**25**
26	0.059 2054	0.062 5674	0.066 0214	0.069 5643	0.076 9043	**26**
27	0.057 8524	0.061 2385	0.064 7195	0.068 2919	0.075 6972	**27**
28	0.056 6026	0.060 0130	0.063 5208	0.067 1225	0.074 5926	**28**
29	0.055 4454	0.058 8799	0.062 4146	0.066 0455	0.073 5796	**29**
30	0.054 3713	0.057 8301	0.061 3915	0.065 0514	0.072 6489	**30**
31	0.053 3724	0.056 8554	0.060 4434	0.064 1321	0.071 7922	**31**
32	0.052 4415	0.055 9486	0.059 5632	0.063 2804	0.071 0023	**32**
33	0.051 5724	0.055 1036	0.058 7445	0.062 4900	0.070 2729	**33**
34	0.050 7597	0.054 3148	0.057 9819	0.061 7554	0.069 5984	**34**
35	0.049 9983	0.053 5773	0.057 2704	0.061 0717	0.068 9739	**35**
36	0.049 2842	0.052 8869	0.056 6058	0.060 4345	0.068 3948	**36**
37	0.048 6132	0.052 2396	0.055 9840	0.059 8398	0.067 8574	**37**
38	0.047 9821	0.051 6319	0.055 4017	0.059 2842	0.067 3581	**38**
39	0.047 3878	0.051 0608	0.054 8557	0.058 7646	0.066 8938	**39**
40	0.046 8273	0.050 5235	0.054 3431	0.058 2782	0.066 4615	**40**
41	0.046 2982	0.050 0174	0.053 8616	0.057 8223	0.066 0589	**41**
42	0.045 7983	0.049 5402	0.053 4087	0.057 3947	0.065 6834	**42**
43	0.045 3254	0.049 0899	0.052 9823	0.056 9933	0.065 3331	**43**
44	0.044 8777	0.048 6645	0.052 5807	0.056 6163	0.065 0061	**44**
45	0.044 4534	0.048 2625	0.052 2020	0.056 2617	0.064 7005	**45**
46	0.044 0511	0.047 8820	0.051 8447	0.055 9282	0.064 4149	**46**
47	0.043 6692	0.047 5219	0.051 5073	0.055 6142	0.064 1477	**47**
48	0.043 3065	0.047 1806	0.051 1886	0.055 3184	0.063 8977	**48**
49	0.042 9617	0.046 8571	0.050 8872	0.055 0396	0.063 6636	**49**
50	0.042 6337	0.046 5502	0.050 6021	0.054 7767	0.063 4443	**50**

V. ANNUITY THAT 1 WILL PURCHASE; $1/a_{\overline{n|}} = i/(1-v^n)$; $1/s_{\overline{n|}} = 1/a_{\overline{n|}} - i$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
51	0.042 3216	0.046 2588	0.050 3323	0.054 5287	0.063 2388	**51**
52	0.042 0243	0.045 9821	0.050 0768	0.054 2945	0.063 0462	**52**
53	0.041 7410	0.045 7191	0.049 8347	0.054 0733	0.062 8655	**53**
54	0.041 4709	0.045 4691	0.049 6052	0.053 8644	0.062 6960	**54**
55	0.041 2132	0.045 2312	0.049 3875	0.053 6669	0.062 5370	**55**
56	0.040 9673	0.045 0049	0.049 1811	0.053 4801	0.062 3876	**56**
57	0.040 7325	0.044 7893	0.048 9851	0.053 3034	0.062 2474	**57**
58	0.040 5081	0.044 5840	0.048 7990	0.053 1363	0.062 1157	**58**
59	0.040 2937	0.044 3884	0.048 6222	0.052 9780	0.061 9920	**59**
60	0.040 0886	0.044 2018	0.048 4543	0.052 8282	0.061 8757	**60**
61	0.039 8925	0.044 0240	0.048 2946	0.052 6823	0.061 7664	**61**
62	0.039 7048	0.043 8543	0.048 1428	0.052 5518	0.061 6637	**62**
63	0.039 5251	0.043 6924	0.047 9985	0.052 4244	0.061 5670	**63**
64	0.039 3531	0.043 5378	0.047 8611	0.052 3037	0.061 4762	**64**
65	0.039 1883	0.043 3902	0.047 7305	0.052 1892	0.061 3907	**65**
66	0.039 0303	0.043 2492	0.047 6061	0.052 0806	0.061 3102	**66**
67	0.038 8789	0.043 1145	0.047 4876	0.051 9776	0.061 2345	**67**
68	0.038 7337	0.042 9858	0.047 3749	0.051 8799	0.061 1633	**68**
69	0.038 5945	0.042 8627	0.047 2675	0.051 7872	0.061 0963	**69**
70	0.038 4609	0.042 7451	0.047 1651	0.051 6992	0.061 0331	**70**
71	0.038 3328	0.042 6325	0.047 0676	0.051 6156	0.060 9737	**71**
72	0.038 2097	0.042 5249	0.046 9747	0.051 5363	0.060 9177	**72**
73	0.038 0916	0.042 4219	0.046 8861	0.051 4610	0.060 8651	**73**
74	0.037 9782	0.042 3233	0.046 8016	0.051 3895	0.060 8154	**74**
75	0.037 8692	0.042 2290	0.046 7210	0.051 3216	0.060 7687	**75**
76	0.037 7645	0.042 1387	0.046 6442	0.051 2571	0.060 7246	**76**
77	0.037 6639	0.042 0522	0.046 5709	0.051 1958	0.060 6831	**77**
78	0.037 5672	0.041 9694	0.046 5010	0.051 1376	0.060 6441	**78**
79	0.037 4743	0.041 8901	0.046 4343	0.051 0822	0.060 6072	**79**
80	0.037 3849	0.041 8141	0.046 3707	0.051 0296	0.060 5725	**80**
81	0.037 2989	0.041 7413	0.046 3099	0.050 9796	0.060 5398	**81**
82	0.037 2163	0.041 6715	0.046 2520	0.050 9321	0.060 5090	**82**
83	0.037 1368	0.041 6046	0.046 1966	0.050 8869	0.060 4800	**83**
84	0.037 0602	0.041 5405	0.046 1438	0.050 8440	0.060 4526	**84**
85	0.036 9866	0.041 4791	0.046 0933	0.050 8032	0.060 4268	**85**
86	0.036 9158	0.041 4202	0.046 0452	0.050 7643	0.060 4024	**86**
87	0.036 8476	0.041 3637	0.045 9992	0.050 7274	0.060 3796	**87**
88	0.036 7819	0.041 3095	0.045 9552	0.050 6923	0.060 3580	**88**
89	0.036 7187	0.041 2576	0.045 9132	0.050 6589	0.060 3376	**89**
90	0.036 6578	0.041 2078	0.045 8732	0.050 6271	0.060 3184	**90**
91	0.036 5992	0.041 1600	0.045 8349	0.050 5969	0.060 3003	**91**
92	0.036 5427	0.041 1141	0.045 7983	0.050 5681	0.060 2832	**92**
93	0.036 4883	0.041 0701	0.045 7633	0.050 5408	0.060 2671	**93**
94	0.036 4359	0.041 0279	0.045 7299	0.050 5148	0.060 2519	**94**
95	0.036 3855	0.040 9874	0.045 6980	0.050 4900	0.060 2376	**95**
96	0.036 3368	0.040 9485	0.045 6675	0.050 4665	0.060 2241	**96**
97	0.036 2899	0.040 9112	0.045 6383	0.050 4441	0.060 2114	**97**
98	0.036 2448	0.040 8754	0.045 6105	0.050 4227	0.060 1994	**98**
99	0.036 2012	0.040 8410	0.045 5838	0.050 4024	0.060 1880	**99**
100	0.036 1593	0.040 8080	0.045 5584	0.050 3831	0.060 1774	**100**

VI. POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	**51**	2 601	132 651	7.141	3.708
2	4	8	1.414	1.259	52	2 704	140 608	7.211	3.732
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.779
5	25	125	2.236	1.709	55	3 025	166 375	7.416	3.802
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.825
7	49	343	2.645	1.912	57	3 249	185 193	7.549	3.848
8	64	512	2.828	2.000	58	3 364	195 112	7.615	3.870
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.892
10	100	1 000	3.162	2.154	**60**	3 600	216 000	7.745	3.914
11	121	1 331	3.316	2.223	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.957
13	169	2 197	3.605	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.741	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.872	2.466	65	4 225	274 625	8.062	4.020
16	256	4 096	4.000	2.519	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.061
18	324	5 832	4.242	2.620	68	4 624	314 432	8.246	4.081
19	361	6 859	4.358	2.668	69	4 761	328 509	8.306	4.101
20	400	8 000	4.472	2.714	**70**	4 900	343 000	8.366	4.121
21	441	9 261	4.582	2.758	71	5 041	357 911	8.426	4.140
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.795	2.843	73	5 329	389 017	8.544	4.179
24	576	13 824	4.898	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.717	4.235
27	729	19 683	5.196	3.000	77	5 929	456 533	8.774	4.254
28	784	21 952	5.291	3.036	78	6 084	474 552	8.831	4.272
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.290
30	900	27 000	5.477	3.107	**80**	6 400	512 000	8.944	4.308
31	961	29 791	5.567	3.141	81	6 561	531 441	9.000	4.326
32	1 024	32 768	5.656	3.174	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.744	3.207	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.830	3.239	84	7 056	592 704	9.165	4.379
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.219	4.396
36	1 296	46 656	6.000	3.301	86	7 396	636 056	9.273	4.414
37	1 369	50 653	6.082	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.361	88	7 744	681 472	9.380	4.447
39	1 521	59 319	6.244	3.391	89	7 921	704 969	9.433	4.464
40	1 600	64 000	6.324	3.419	**90**	8 100	729 000	9.486	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.497
42	1 764	74 088	6.480	3.476	92	8 464	778 688	9.591	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.643	4.530
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.546
45	2 025	91 125	6.708	3.556	95	9 025	857 375	9.746	4.562
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.797	4.578
47	2 209	103 823	6.855	3.608	97	9 409	912 673	9.848	4.594
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.949	4.626
50	2 500	125 000	7.071	3.684	**100**	10 000	1 000 000	10.000	4.641

VII. LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	*380	*766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	**38	*407	*776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	**26
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	**12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	*126	*413	*699	*985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	**51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	**29	*303	*577	*850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	**51	*319	*586	*853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	**50	*300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
186	9513	9746	9980	*213	*446	*679	*912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	278754	8982	9211	9439	9667	9895	*123	*351	*578	*806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	*161	*378	*595	*813

N.	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	**56	*268	*481	*693	*906	1118	1330	1542
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	***8	*211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	**47	*246
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
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223	8305	8500	8694	8889	9083	9278	9472	9666	9860	**54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
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230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	*143	*328	*513	*698	*883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	**30
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	**51	*228	*405	*582	*759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
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252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	*102	*271	*440	*609	*777	*946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474
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262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	*121	*286	*451	*616	*781	*945	1110	1275	1439
264	421604	1788	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	**75	*236	*398	*559	*720	*881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
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273	6163	6322	6481	6640	6798	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	*122	*279	*437	*594	*752
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
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279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	**95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	*146	*296	*447	*597	*748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	462398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	*116	*263	*410	*557	*704	*851	*998	1145
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
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302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
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309	9958	**99	*239	*380	*520	*661	*801	*941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	**99	*236	*374	*511	*648	*785	*922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
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323	9203	9337	9471	9606	9740	9874	***9	*143	*277	*411
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326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
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329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
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331	9828	9959	**90	*221	*353	*484	*615	*745	*876	1007
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333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
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339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
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342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
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346	9076	9202	9327	9452	9578	9703	9829	9954	**79	*204
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348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
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352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	*106
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328
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357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
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363	9907	**26	*146	*265	*385	*504	*624	*743	*863	*982
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
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374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
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380	579784	9898	**12	*126	*241	*355	*469	*583	*697	*811
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382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
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389	9950	**61	*173	*284	*396	*507	*619	*730	*842	*953
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
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394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
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398	9883	9992	*101	*210	*319	*428	*537	*646	*755	*864
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402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
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407	9594	9701	9808	9914	**21	*128	*234	*341	*447	*554
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409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
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414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	**32
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	623249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
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430	633468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
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436	9486	9586	9686	9785	9885	9984	**84	*183	*283	*382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
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440	643453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
446	9335	9432	9530	9627	9724	9821	9919	**16	*113	*210
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116

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451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
457	9916	**11	*106	*201	*296	*391	*486	*581	*676	*771
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
467	9317	9410	9503	9596	9689	9782	9875	9967	**60	*153
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6068	7059	7151	7242	7333	7424	7516
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
478	9428	9519	9610	9700	9791	9882	9973	**63	*154	*245
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	681241	1332	1422	1513	1603	1693	1784	1874	1964	2055
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
489	9309	9398	9486	9575	9664	9753	9841	9930	**19	*107
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883

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502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	707570	7655	7740	7826	7911	7996	8081	8166	8251	8336
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	**33
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	716003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	**77
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374
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530	724276	4358	4440	4522	4604	4685	4767	4849	4931	5013
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
537	9974	**55	*136	*217	*298	*378	*459	*540	*621	*702
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
549	9572	9651	9731	9810	9889	9968	**47	*126	*205	*284

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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
562	9736	9814	9891	9968	**45	*123	*200	*277	*354	*431
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
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571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
575	9668	9743	9819	9894	9970	**45	*121	*196	*272	*347
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
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580	763428	3503	3578	3653	3727	3802	3877	3952	4027	4101
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	**42
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	770852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
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602	9596	9669	9741	9813	9885	9957	**29	*101	*173	*245
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
616	9581	9651	9722	9792	9863	9933	***4	**74	*144	*215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022
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622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492
645	9560	9627	9694	9762	9829	9896	9964	**31	**98	*165
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847

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650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478
660	819544	9610	9676	9741	9807	9873	9939	***4	**70	*136
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	826075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
676	9947	**11	**75	*139	*204	*268	*332	*396	*460	*525
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
680	832509	2573	2637	2700	2764	2828	2892	2956	3020	3083
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	**43
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
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702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	851258	1320	1381	1442	1503	1564	1625	1686	1747	1809
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
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725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
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727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	863323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
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734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	869232	9290	9349	9408	9466	9525	9584	9642	9701	9760
741	9818	9877	9935	9994	**53	*111	*170	*228	*287	*345
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
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752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
758	9669	9726	9784	9841	9898	9956	**13	**70	*127	*185
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756
760	880814	0871	0928	0985	1042	1099	1156	1213	1271	1328
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
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767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
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771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561
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773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
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776	9862	9918	9974	**30	**86	*141	*197	*253	*309	*365
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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794	9821	9875	9930	9985	**39	**94	*149	*203	*258	*312
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967
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812	9556	9610	9663	9716	9770	9823	9877	9930	9984	**37
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
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817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
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819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
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830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549
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835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
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838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
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840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744
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842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
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844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
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847	7883	7935	7986	8037	8088	8140	9191	8242	8293	8345
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
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852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
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869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
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871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
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877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
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891	9878	9926	9975	**24	**73	*121	*170	*219	*267	*316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194

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902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	**42	**90	*138	*185	*233	*280	*328	*376	*423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	**21	**68	*114	*161	*207	*254	*300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678

N.	0	1	2	3	4	5	6	7	8	9
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	**28	**72	*117	*161	*206	*250	*294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

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